STAT31440 Applied Analysis

Notes for Exam Preparations

Seung Chul Lee

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Contents

1 Definitions

1.1 Metric Space

- $\limsup/\liminf$ of sets

$$\limsup_{i\to\infty} A_i = \bigcap_{j=1}^{\infty}\left(\bigcup_{i=j}^{\infty} A_i\right), \quad \liminf_{i\to\infty} A_i = \bigcup_{j=1}^{\infty}\left(\bigcap_{i=j}^{\infty} A_i\right)$$

- Metric space
  A metric space (X, d) consists of a non-empty set X, $d : X \times X \to [0,\infty)$ s.t.
  1. $d(x,y) = d(y,x), \forall x, y \in X$ (symmetry)
  2. $d(x,y) = 0 \Rightarrow x = y, \forall x, y \in X$.
  3. $d(x,z) \le d(x,y) + d(y,z), \forall x, y, z \in X$ (triangle inequality).
- Diameter
  (X, d): metric space, $A \subset X$, then

$$\operatorname{diam} A := \begin{cases} \sup_{x,y\in A} d(x,y) & , A \neq \phi \\ 0 & , A = \phi \end{cases}$$

  and we say A is bounded if $\operatorname{diam} A < \infty$.

- Normed linear space
  Let E be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that E is a normed linear space if $\exists \|\cdot\| : E \to [0, \infty)$ s.t.
  1. $\|x\| = 0 \iff x = 0 \in E, \forall x \in E$.
  2. $\|\alpha x\| = |\alpha| \|x\|, \forall x \in E, \alpha \in \mathbb{F}$.
  3. $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E$.
- ℓ^p space
$$\ell^p(\mathbb{N}^*) := \left\{ (x_1, \dots, x_n, \dots) : x_i \in \mathbb{R}, \forall i \text{ and } \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} < \infty \right\}$$
- p-norm
$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$$
- Convergence of sequences
  $(x_i)_{i \geq 1} \subset X$, a sequence, converges to $x_* \in X$ if $\forall \varepsilon > 0, \exists n \geq 1$ s.t. $\forall i \geq n, d(x_i, x_*) < \varepsilon$.
- Cauchy sequence
  $(x_i) \subset X$ a sequence in X is a Cauchy sequence if for all $\varepsilon > 0$, $\exists m \geq 1$ for all $i, j \geq m$, $d(x_i, x_j) < \varepsilon$.
- Complete metric space
  A metric space (X, d) is complete if every Cauchy sequence in X converges in X.
- Banach space
  If a normed linear space E is complete w.r.t. the metric $d(x, y) = \|x - y\|$, then $(E, \|\cdot\|)$ is called a Banach space.
- Convergence of series
  If $(S_n)_{n \geq 1}$ defined as $S_n := \sum_{i=1}^{n} x_i$ converges to $s \in \mathbb{R}$, $\sum_{i=1}^{\infty} x_i$ is said to converge to s.
- Absolute convergence of series
  $\sum_{i=1}^{\infty} x_i$ is said to be absolutely convergent if $\sum_{i=1}^{\infty} |x_i|$ converges in $\mathbb{R}$.
- Upper/lower bound
  $A \subset \mathbb{R}$ has an upper bound $M \in \mathbb{R}$, lower bound $L \in \mathbb{R}$ if $x \in A \Rightarrow x \leq M$, $x \in A \Rightarrow x \geq L$ and A is said to be bounded from above (below) if such an M (L) exists.
- Supremum/infimum
  An upper bound M for a set $A \subset \mathbb{R}$ is a least upper bound (supremum) if $M \leq M'$ for all upper bounds M' of A. Similarly, a lower bound L of a set $A \subset \mathbb{R}$ is a greatest lower bound (infimum) if $L \geq L'$ for all lower bounds L' of A.
- $\limsup / \liminf$
$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup\{x_k : k \geq n\} = \inf\{\sup\{x_k : k \geq n\} : n \geq 1\}$$
$$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf\{x_k : k \geq n\} = \sup\{\inf\{x_k : k \geq n\} : n \geq 1\}$$
- Continuity
  $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \mathbb{R}, |x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.
- Uniform continuity
  $f : X \to Y$ is uniformly continuous on X if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in X, d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

- Sequential continuity
  X, Y: metric spaces. $f : X \to Y$ is sequentially continuous at $x \in X$ if $\forall (x_n)_{n\geq 1} \subset X$ s.t. $x_n \to x$ as $n \to \infty$, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x) \in Y$ as $n \to \infty$.
- Upper/Lower semicontinuity
  A function $f : X \to \mathbb{R}$ is upper semicontinuous on X if $\forall (x_n)_{n\geq 1} \subset X$ such that $x_n \to x$ for some $x \in X$ implies $f(x) \geq \limsup_{n\to\infty} f(x_n)$.
  Similarly, $f : X \to \mathbb{R}$ is lower semicontinuous on X if $\forall (x_n)_{n\geq 1} \subset X$, $x_n \to x, x \in X$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$.
- Open/Closed ball
  The open ball $B_r(x) = B(x; r)$ is the set $B_r(x) := \{y \in X : d(x, y) < r\}$ and closed ball $\overline{B_r}(x) := \{y \in X : d(x, y) \leq r\}$.
- Open/Closed sets
  $G \subset X$ is an open set if for every $x \in G$, $\exists r > 0$ s.t. $B_r(x) \subset G$. A set $F \subset X$ is closed in X if $X \setminus F$ is open.
- Topology on a set
  τ is a topology on X if the family τ of open subsets of X satisfies
  1. $\phi, X \in \tau$.
  2. $A, B \in \tau \Rightarrow A \cap B \in \tau$.
  3. $\{A_i : i \in I$ an arbitrary family of elements of $\tau\} \Rightarrow \bigcup_{i\in I} A_i \in \tau$.

  X equipped with τ is called a topological space.
- Convergence in topological space
  $(x_n)_{n\geq 1} \subset X$ converges to $x \in X$ for a topological space (X, τ) if for all $A \in \tau$ with $x \in A$, $\exists N \geq 1$ s.t. $\forall n \geq N, x_n \in A$.
- Measure zero (child's version)
  $A \subset \mathbb{R}$ is said to have measure zero if for every $\varepsilon > 0$ there is a countable collection of open intervals (I_n) s.t. $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{i=1}^{\infty} length(I_n) < \varepsilon$.
- Closure
  The closure of a set A in a metric (or topological) space X is
  $$\overline{A} = \bigcap_{A\subset F\subset X, F:\text{closed}} F$$
  which is the smallest closed set containing A.
- Dense in a metric space
  (X, d): metric space. $A \subset X$ is dense in X if $\overline{A} = X$.
- Separable
  (X, d): metric space is separable if X contains a countable dense subset.
- Isometry/Isomorphism
  X, Y: metric space. $i : X \to Y$ is an isometry if
  $$d(i(x_1), i(x_2)) = d(x_1, x_2), \forall x_1, x_2 \in X.$$
  If i is an isometry that is surjective (onto), it is a (metric space) isomorphism.
- Completion of a metric space
  Given metric space (X, d), another metric space $(\tilde{X}, \tilde{d})$ is a completion of X if
  1. $\exists i : X \to \tilde{X}$ an isometry.
  2. $i(X)$ is dense in $\tilde{X}$.
  3. $(\tilde{X}, \tilde{d})$ is complete.

- Equivalence relation
  A relation $\sim$ defines an equivalence relation if it is
  1. reflexive: $a \sim a, \forall a$
  2. symmetric: $a \sim b \iff b \sim a, \forall a, b$
  3. transitive: $a \sim b, b \sim c \Rightarrow a \sim c, \forall a, b, c$
- Sequential compactness
  X: metric space. $K \subset X$ is sequentially compact if every sequence in K has a subsequence which converges to a point in K.
- Open cover
  X: metric space, $A \subset X$. A collection $\{G_\alpha\}_{\alpha \in I}$ of subsets of X is said to cover A if
  $$A \subset \bigcup_{\alpha \in I} G_\alpha$$
  If every G_α is open, we say $\{G_\alpha\}$ is an open cover of A.
- ε-net
  For $\varepsilon > 0$ and $A \subset X$, a subset $E = \{x_\alpha : \alpha \in I\}$, with I: arbitrary index set, is a ε-net for A if $\{B_\varepsilon(x_\alpha) : \alpha \in I\}$ is an open cover of A, i.e., $A \subset \bigcup_{\alpha \in I} B_\varepsilon(x_\alpha)$.
  If I is finite and E is an ε-net, then E is a finite ε-net.
- Totally bounded
  X: metric space. $A \subset X$ is totally bounded if for every $\varepsilon > 0$ there exists a finite ε-net for A.
- Compactness
  X: metric space. $K \subset X$ is compact if every open cover of K has a finite subcover.
- Precompact:
  X: metric space. $A \subset X$ is precompact if $\overline{A}$ is compact.

1.2 Space of Continuous Functions/Contraction Mapping Theorem

- Some subspaces:
  - $C(X)$: space of real-valued continuous functions in X.
  - $C_b(X)$: space of real-valued bounded continuous functions in X.
  - $C_c(X)$: space of real-valued continuous functions in X with compact support, i.e.,
    $$C_c(X) := \{f : X \to \mathbb{R} : f \in C(X), \ \operatorname{supp} f \subset X : \ \text{compact}\}$$
  - $C_0(X)$: closure of $C_c(X)$ in $C_b(X)$.
  $$C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X)$$
  with equality if X: compact.
- Pointwise convergence:
  $f_n \to f$ pointwise as $n \to \infty$ if for all $x \in X$, $f_n(x) \to f(x)$ (in $\mathbb{R}$) as $n \to \infty$.
- Uniform norm:
  Uniform norm $\|\cdot\|_\infty$ is a norm defined on $C_b(X)$ as
  $$\|f\|_\infty = \sup_{x \in X} |f(x)|$$
  and convergence in the uniform norm is called uniform convergence.
- Equicontinuity:
  X: metric space. A family $\mathscr{F}$ of functions in $C(X)$ is equicontinuous if $\forall x \in X, \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall y \in X$, $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathscr{F}$.

- Lipschitz continuity:
  (X, d): metric space. $f : X \to \mathbb{R}$ is said to be Lipschitz continuous on X if $\exists L > 0$ s.t. $|f(x) - f(y)| \le L \cdot d(x, y), \forall x, y \in X$.

- Lipschitz constant:
$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} = \inf\{C : |f(x) - f(y)| \le C \cdot d(x, y), \forall x, y \in X\}$$

- Space of continuously differentiable functions:
$$C^1(X) = \{f \in C(X) : f' \text{ is continuous on } X\}.$$

- ODE IVP Solution:
$$(*) \begin{cases} \frac{d}{dt} u(t) = f(t, u(t)), t \ge 0 \\ u(0) = u_0 \end{cases}$$
  A solution to $(*)$ is a function $u \in C^1(I)$, $I \subset \mathbb{R}$ open with $0 \in I$ s.t. $\dot{u}(t) = f(t, u(t)), \ \ \forall t \in I$.

- Contraction:
  (X, d): metric space. A map $T : X \to X$ is a contraction if $\exists 0 \le c < 1$ s.t.
$$d(T(x), T(y)) \le C \cdot d(x, y), \ \ \forall x, y \in X.$$

- Bounded linear map:
  A linear map $A : X \to Y$ is bounded if $\exists c > 0$ s.t. $\|Ax\| \le c\|x\|, \ \ \forall x \in X$.

- Differentiable:
  X, Y: Banach spaces, $f : X \to Y$. f is differentiable at $x \in X$ if $\exists f' : X \to Y$ linear bounded map s.t.
$$f(x + \varepsilon h) = f(x) + \varepsilon f'(x)h + o(\varepsilon)$$
  as $\varepsilon \to 0$.

1.3 Measure and Integration

- σ-algebra:
  A σ-algebra on a set X is a collection $\mathcal{A}$ of subsets of X s.t.
  1. $\phi \in \mathcal{A}$
  2. $A \in \mathcal{A} \Rightarrow A^c = X \setminus A \in \mathcal{A}$
  3. If $(A_i)_{i \in I}$ is a countable family of sets in $\mathcal{A}$, then $\bigcup_{i \in I} A_i \in \mathcal{A}$.

- Measurable space:
  A set X with a σ-algebra $\mathcal{A}$ on X is a measurable space. The elements of $\mathcal{A}$ are measurable sets.

- Measure:
  $(X, \mathcal{A})$: measurable space. A measure on $(X, \mathcal{A})$ is a map $\mu : A \to [0, \infty]$ s.t.
  1. $\mu(\phi) = 0$
  2. If $(A_i)_{i \in I}$ is a countable family of pairwise disjoint sets in $\mathcal{A}$, then
$$\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i).$$

- Finite/σ-finite measures:
  μ is finite if $\mu(X) < \infty$, and σ-finite if $\exists A_1, A_2, \ldots$ s.t.
$$X = \bigcup_{i=1}^{\infty} A_i$$
  with $\mu(A_i) < \infty, \ \ \forall i = 1, 2, \ldots$.

- Measure space:
  $(X, \mathcal{A}, \mu)$ is a measure space with $\mathcal{A}$ a σ-algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ a measure.
- Generated σ-algebra:
  Given $\mathcal{F} \subset \mathcal{P}(X)$, the σ-algebra generated by $\mathcal{F}$, $\mathcal{A}(\mathcal{F})$, is the intersection of all σ-algebras containing $\mathcal{F}$. This is the smallest σ-algebra that contains $\mathcal{F}$.
- Borel σ-algebra:
  (X, τ): topological space. The Borel σ-algebra, denoted $\mathcal{B}(X)$, is $\mathcal{A}(\tau)$, the σ-algebra generated by the collection of open sets.
- Counting measure:
  X: any set, $\mathcal{A} = \mathcal{P}(X)$, the counting measure ν is given by $\nu : \mathcal{P} \to [0, \infty]$, $\nu(A) = \#A$ (i.e., the number of elements) for $A \subset X$.
- Dirac delta measure:
  Given $x_0 \in \mathbb{R}^n$, the Dirac delta measure

$$\delta_{x_0} : \mathcal{B}(\mathbb{R}^n) \to [0, \infty], \;\; \delta_{x_0}(A) = \begin{cases} 0, & \text{if } x_0 \notin A \\ 1, & \text{if } x_0 \in A \end{cases}$$

- Lebesgue measure:
  The unique measure λ on $\mathcal{B}(\mathbb{R}^n)$ satisfying

$$\lambda((a_1, b_1) \times \cdots \times (a_n, b_n)) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

  is the Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$.
- Complete measure space:
  $(X, \mathcal{A}, \mu)$, a measure space, is complete if every subset of a set of measure zero is measurable.
- Lebesgue measurable sets:
  The class $\mathscr{L}(\mathbb{R}^n)$ of Lebesgue measurable sets is the completion of $\mathcal{B}(\mathbb{R}^n)$ w.r.t. the Lebesgue measure.
- Measure zero:
  $(X, \mathcal{A}, \mu)$: measure space. $A \subset X$ has measure zero if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- Almost everywhere (a.e.):
  A property that holds except on a set of measure zero is said to hold almost everywhere (a.e.).
- Essential supremum:
  $\operatorname{ess\,sup} A = \inf\{C : \exists N \subset \mathbb{R} \text{ measure zero s.t. } x \le c, \forall x \in A \setminus N\}$.
- Equality a.e. of functions:
  $(X, \mathcal{A}, \mu)$: measure space. f, g: measurable functions. f and g are equal a.e. w.r.t. μ if $\mu\left(\{x \in X : f(x) \neq g(x)\}\right) = 0$.
- Measurable function (w.r.t. measurable spaces):
  $(X, \mathcal{A})$, $(Y, \mathcal{B})$: measurable spaces, $f : X \to Y$ is $(X, \mathcal{A})$–$(Y, \mathcal{B})$ measurable if $f^{-1}(B) \in \mathcal{A}$, $\forall B \in \mathcal{B}$.
- Extended real line:
  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with conventions $0 \cdot \infty = 0 \cdot (-\infty) = 0$.
- Measure preserving:
  $T : X \to X$ measurable, $(X, \mathcal{A}, \mu)$: measure space. T is measure preserving if

$$\mu(T^{-1}(A)) = \mu(A), \forall A \in \mathcal{A}.$$

- Measurable function:
  $f : X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) is measurable if it is $(X, \mathcal{A}, \mu)$–$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable.

- Characteristic (or indicator) function:
  $(X, \mathcal{A})$: measurable space, $A \in \mathcal{A}$,

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$$

- Simple function:
  $\varphi : X \to \mathbb{R}$ written as

$$\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$$

  for $c_1, \ldots, c_n \in \mathbb{R}$, $A_1, \ldots, A_n \in \mathcal{A}$ is a simple function.

- Partitioning a function:
  $f : X \to \overline{\mathbb{R}}$, measurable.

$$f = f_+ - f_-, \text{ where } f_+ = \max\{f, 0\},\ f_- = -\min\{f, 0\}$$

- Lebesgue integral (of a simple function):
  $(X, \mathcal{A}, \mu)$: measure space. For $\varphi : X \to \overline{\mathbb{R}}$, a simple measurable function, $\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$, the Lebesgue integral is

$$\int \varphi d\mu = \sum_{i=1}^{n} c_i \mu(A_i)$$

- Lebesgue integral (of a general function):
  $(X, \mathcal{A}, \mu)$: measure space. $f : X \to [0, \infty]$ measurable.

$$\int f d\mu = \sup \left\{ \int \varphi d\mu : \varphi \text{ simple with } \varphi \leq f \right\}$$

- Integrable:
  A function f is integrable (summable) if $\int |f| d\mu < \infty$.

- Product σ-algebra:
  $(X, \mathcal{A}), (Y, \mathcal{B})$: measurable spaces. The product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is the σ-algebra generated by the collection of sets $\xi = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

- L^p space:
  $(X, \mathcal{A}, \mu)$: measure space. $1 \leq p < \infty$.

$$L^p(X) = \left\{ \text{equivalence classes of functions } f : X \to \mathbb{C} : \int |f|^p d\mu < \infty \right\}$$

  Also,

$$L^\infty(X) = \left\{ \text{equivalence classes of functions } f : X \to \mathbb{C} : \operatorname*{ess\,sup}_{x \in X} |f(x)| < \infty \right\}$$

- L^p norm:
  For $1 \leq p < \infty$,

$$\|f\|_{L^p(X)} = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

  For $p = \infty$,

$$\|f\|_{L^\infty(X)} = \operatorname*{ess\,sup}_{x \in X} |f(x)|$$

2 Useful Facts

2.1 Metric Space

- Cauchy-Schwarz inequality
$$x \cdot y \leq \|x\|\|y\|$$
- $(X, d_X), (Y, d_Y)$: two metric spaces
$\Rightarrow X \times Y$ is also a metric space with product metric defined as
$$d(u,v) = d_X(u_X, v_X) + d_Y(u_Y, v_Y), u = (u_X, u_Y) \in X \times Y, v = (v_X, v_Y) \in X \times Y$$
- If E: normed linear space, then E is a metric space with $d(x,y) = \|x - y\|, \forall x, y \in E$, i.e., the norm-induced metric.
- E: normed linear space, $(x_n)_{n\geq 1}$: a sequence in E.
If $x_n \to x \in E$ for some $x \in E$, then
$$\lim_{n\to\infty} \|x_n\| = \|x\|$$
i.e., $\|\cdot\|$ is continuous on E.
- Hölder's inequality (in $\mathbb{R}^n$)
$x, y \in \mathbb{R}^n$, $1 \leq p < \infty, 1 \leq q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ (conjugate exponents). Then,
$$\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q.$$
- (x_n): Cauchy $\Rightarrow$ (x_n): bounded.
- (x_n): converges $\Rightarrow$ (x_n): Cauchy.
- A normed linear space may be equipped with multiple different norms.
- $\sum x_n$: absolutely convergent $\Rightarrow$ $\sum x_n$: convergent
- In $\mathbb{R}$ or a Banach space:
$$\sum x_i \to s \iff \forall \varepsilon > 0, \exists N \geq 1 \text{ s.t. } |x_{n+1} + \cdots + x_{n+p}| < \varepsilon, \forall n \geq N, p \geq 1$$
(a version of Cauchy criterion).
- Existence of inf/sup for bounded sets $A \subset \mathbb{R} \iff$ completeness of $\mathbb{R}$.
- $\liminf$, $\limsup$ always defined for sequences in $\mathbb{R}$.
$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$
- $$x_n \to x \iff \liminf_{n\to\infty} = \limsup_{n\to\infty} = x$$
- $[a,b]$: closed, bounded interval in $\mathbb{R}$, f: continuous on $[a,b] \Rightarrow f$: uniformly continuous on $[a,b]$.
- $f : \mathbb{R}^n \to \mathbb{R}^n$ is affine if
$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \quad \forall x, y \in \mathbb{R}^n, 0 \leq t \leq 1$$
Every affine function is uniformly continuous on $\mathbb{R}^n$ and can be written as $f : x \mapsto Ax + b$ for $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n), b \in \mathbb{R}^n$.
- X, Y: metric spaces. $f : X \to Y, x \in X, f$: continuous at $x \Rightarrow f$: sequentially continuous at x.

- f: continuous on $X \iff f$: both upper and lower semicontinuous.
- X, Y: metric spaces, $f : X \to Y$, continuous on $X \iff$ for every $G \subset Y$ open, $f^{-1}(G)$ is open in X.
- Open mapping theorem
  E, F: Banach spaces, $T : E \to F$, continuous, linear, surjective $\Rightarrow T$: open map, i.e., maps open sets to open sets.
- E, F : Banach spaces, $T : E \to F$, continuous, linear, bijective $\Rightarrow T^{-1}$: continuous.
- f: continuous on $X \iff \forall F \subset Y$ closed $f^{-1}(F) \subset X$ is closed.
- Finite unions of closed sets are closed; arbitrary intersections of closed sets are closed.
- Infinite intersections of open sets may not be open; infinite unions of closed sets may not be closed.
- Every open set $G \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals.
- (X, d): metric space. $F \subset X$: closed $\iff$ for every sequence $(x_n) \subset X$ convergent in X, if $x_n \in F$ for all $n \geq 1$, then $\lim_{n\to\infty} x_n \in F$.
- (X, d): complete metric space, $F \subset X$ is a complete metric space (w.r.t. induced metric space) $\iff F$ is a closed set in X.
- Sequential equivalent of closure

$$\overline{A} = \{x \in X : \exists (a_n)_{n\geq 1} \subset A, a_n \to x\}.$$

- Any isometry $i : X \to Y$ is injective.
- Uniqueness of completion
  (X, d): metric space. If $(\tilde{X}_1, \tilde{d}_1), (\tilde{X}_2, \tilde{d}_2)$ are two completions of X, then they are isomorphic.
- Equivalence class of Cauchy sequences

$$(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0, n \to \infty$$

- Bolzano-Weierstrass Theorem
  Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- Heine-Borel Theorem
  $A \subset \mathbb{R}^n$: sequentially compact $\iff A$: closed, bounded.
- Theorem:
  (X, d): metric space. X: sequentially compact $\iff X$: complete and totally bounded.
- Theorem:
  X: metric space. $K \subset X$: sequentially compact $\iff K$: compact.
- Lemma:
  X: metric space, $K \subset X$: sequentially compact. If $\{G_\alpha\}_{\alpha\in I}$ is an open cover of K, then there exists $\delta > 0$ s.t. $\forall A \subset K$, $\mathrm{diam}(A) \leq \delta$ implies $A \subset G_\alpha$ for some $\alpha \in I$.
- $A \subset \mathbb{R}^n$: precompact $\iff A$: bounded.
- X: metric space, $A \subset X$: precompact $\iff$ every sequence in A has a subsequence which converges to some point in X.
- X: metric space, $K \subset X$: compact $\iff K$: closed and precompact.
- X: complete metric space, $A \subset X$: precompact $\Rightarrow A$: totally bounded.
- Theorem:
  K: compact metric space, Y: metric space, $f : X \to Y$ continuous on $K \Rightarrow f(K) \subset Y$: compact.

- Theorem:
  K: compact metric space, Y: metric space, $f : X \to Y$ continuous on $K \Rightarrow f$: uniformly continuous on K.

- Theorem:
  K: compact metric space, $f : K \to \mathbb{R}$ continuous. Then, $\exists x, y \in K$ s.t.

$$f(x) = \inf_{z \in K} f(z), f(y) = \sup_{z \in K} f(z).$$

- Theorem: (Equivalence of norms)
  E: finite dimensional vector space, $\|\cdot\|, \|\cdot\|'$: norms. Then, $\exists c, C > 0$ s.t.

$$c\|x\| \leq \|x\|' \leq C\|x\|.$$

2.2 Space of Continuous Functions/Contraction Mapping Theorem

- Uniform convergence $\Rightarrow$ pointwise convergence.
- (X, d): metric space, $(f_n)_{n \geq 1} \subset C_b(X)$ a sequence. If $f_n \to f$ uniformly for some bounded $f : X \to \mathbb{R}$, then f is continuous.
- X: compact metric space. Then, $C(X)$ is complete, i.e., a Banach space.
- $f : \mathbb{R}^n \to \mathbb{R}$ has compact support $\iff \exists R > 0$ s.t. $f(x) = 0, \forall x \in \mathbb{R}^n$ with $|x| > R$.
- **Ascoli-Arzela thm**:
  If X: compact metric space, then $\mathscr{F} \subset C(X)$ is precompact in $C(X)$ iff it is bounded in $C(X)$ and equicontinuous.
  If X: compact metric space, then $\mathscr{F} \subset C(X)$ is compact in $C(X)$ iff $\mathscr{F}$ is closed, bounded, and equicontinuous.
- f: Lipschitz $\Rightarrow f$: uniformly continuous (but not vice versa).
- $C \subset \mathbb{R}^n$: open, convex, $f : C \to \mathbb{R}$ continuously differentiable on C.
  If the partial derivatives of f and bounded on C, then $\forall x, y \in C$,

$$|f(x) - f(y)| \leq \left(\sup_{y \in C} |\nabla f(x)| \right) |x - y|$$

- (K, d): compact metric space, $M > 0$. Then,
  $\mathscr{F}_M = \{f \in C(K) : f \text{ Lipschitz on } K, \ \operatorname{Lip}(f) \leq M\}$ is an equicontinuous subset of $C(K)$.
  $\mathscr{F}_M$ is also closed.
- Any bounded family of continuously differentiable functions on $C(K)$ with bounded derivatives is precompact in $C(K)$.
- If K: compact metric space, $x_0 \in K$, then

$$\mathscr{B}_M = \{f \in \mathscr{F}_M : f(x_0) = 0\}$$

  is a closed bounded subset of $\mathscr{F}_M$ and thus compact.

- Theorem:
  $f : (t, u) \mapsto f(t, u)$ continuous on $\mathbb{R}^2$. Then, $\forall (t_0, u_0) \in \mathbb{R}^2$, $\exists I \subset \mathbb{R}$ an open interval with $t_0 \in I$ s.t. the IVP

$$\begin{cases} \dot{u} = f(t, u) \\ u(t_0) = u_0 \end{cases}$$

  has a solution $u \in C^1(I)$ on I.

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous in $R = \{(t, u) : |t - t_0| \le T, |u - u_0| \le L\}$ with $|f(t, u)| \le M$ for $(t, u) \in \mathbb{R}^2$, and that $u \mapsto f(t, u)$ is Lipschitz uniformly in t in the sense that $\exists C > 0$ s.t. $|f(t, u) - f(t, v)| \le C|u - v|, \forall (t, u), (t, v) \in R$.
  Set $\delta := \min\{T, L/M\}$. Then, every solution u to $\dot{u}(t) = f(t, u), u(t_0) = u_0$ satisfies
  $$|u(t) - u_0| \le L \text{ for } |t - t_0| < \delta,$$
  and the solution is unique on $|t - t_0| < \delta$.
- Gronwall's inequality:
  $T > 0, u, \varphi \in C([0, T])$ with $u(t) \ge 0$ and $\varphi(t) \ge 0, \forall t \in [0, T]$. Fix $u_0 \ge 0$. If u satisfies
  $$u(t) \le u_0 + \int_0^t \varphi(s)u(s)ds, \quad t \in [0, T],$$
  then
  $$u(t) \le u_0 \exp\left(\int_0^t \varphi(s)ds\right)$$
  for $0 \le t \le T$.
- (X, d): metric space. $T : X \to X$ contraction $\Rightarrow T$: uniformly continuous on X.
- **Banach contraction mapping thm:**
  (X, d): complete metric space, $T : X \to X$ contraction $\Rightarrow T$ has a unique fixed point, i.e., $\exists x \in X$ s.t. $T(x) = x$ and there is exactly one such value x.
- Theorem:
  $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ continuous in t, globally Lipshitz u, uniform in t, $(t, u) \mapsto f(t, u)$.
  $$\begin{cases} \dot{u} = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$
  $I \subset \mathbb{R}$ s.t. $t_0 \in I$. Then, $\exists!$ continuously differentiable u solving the IVP.
- **Inverse Function Thm:**
  $U \subset \mathbb{R}^n$: open, $f : U \to \mathbb{R}^n$, continuously differentiable. $x_0 \in U$ s.t. $(Df)(x_0) \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ is nonsingular. Then, $\exists$ open sets $V \subset U$ containing x_0, $W \subset \mathbb{R}^n$ containing $y_0 = f(x_0)$ and $g : W \to V$ s.t. $g(f(x)) = x$ for $x \in V$ and $f(g(y)) = y$ for $y \in W$. Moreover, $Dg(y) = [(Df)(x)]^{-1}$.
- **Implicit Function Thm:**
  $m, n \ge 1$, $A \subset \mathbb{R}^{n+m}$ open, $F : A \to \mathbb{R}^m$ continuously differentiable. If $(x_0, y_0) \in A$ s.t. $F(x_0, y_0) = 0$ and $(D_y F)(x_0, y_0)$ is invertible, then $\exists$ open sets $W \subset \mathbb{R}^n$, $x_0 \in U$, $V \subset A \subset \mathbb{R}^{n+m}$, $(x_0, y_0) \in V$, and $G : W \to \mathbb{R}^m$ differentiable at x_0 s.t.
  $$\{(x, y) \in V : F(x, y) = 0\} = \{(x, G(x)) : x \in W\}$$

2.3 Measure and Integration

- Lemma:
  $\mathcal{F} \subset \mathcal{P}(X) \Rightarrow \mathcal{A}(\mathcal{F})$ is a σ-algebra.
- Theorem:
  X: set, $\mathcal{F} \subset \mathcal{P}(X)$, $\mathcal{A} = \mathcal{A}(\mathcal{F})$, $\mu : \mathcal{A} \to [0, \infty]$ a measure. If there is a countable family $(A_i)_{i \in I}$ of sets in $\mathcal{F}$ s.t. $\mu(A_i) < \infty$, $\forall i$ and $X = \bigcup_{i \in I} A_i$, then for any measure $\nu : \mathcal{A} \to [0, \infty]$, if $\mu(A) = \nu(A)$, $\forall A \in \mathcal{F}$, then $\mu = \nu$ on $\mathcal{A}$.
- Properties of Borel σ-algebra:
  1. $\mathcal{B}(\mathbb{R})$ is also generated by the collection of open intervals $(a, b) \subset \mathbb{R}$ and the collection of half-open intervals $(a, b] \subset \mathbb{R}$.
  2. $\mathcal{B}(\mathbb{R}^n)$ is generated by family of rectangles
     $$Q = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$$

3. $\mathcal{B}(\mathbb{R}^n)$ is not complete w.r.t. the Lebesgue measure.

- Theorem:
  $A \subset \mathbb{R}^n$: Lebesgue measurable $\iff \forall \varepsilon > 0, \exists F$: closed in $\mathbb{R}^n$, G: open in $\mathbb{R}^n$ s.t. $F \subset A \subset G$ and $\lambda(G \setminus F) < \varepsilon$.
  Moreover, for all such A, $\lambda(A) = \inf\{\lambda(U) : U \text{ open}, A \subset U\} = \sup\{\lambda(K) : K \text{ compact}, K \subset A\}$.
- Properties of Lebesgue measure:
  1. Translation invariant: $A \in \mathscr{L}(\mathbb{R}^n), h \in \mathbb{R}^n \Rightarrow \lambda(\tau_h A) = \lambda(A)$ with $\tau_h A = \{x + h : x \in A\}$.
  2. $T : \mathbb{R}^n \to \mathbb{R}^n$ linear
  $$T(A) = \{Tx : x \in A\}, \;\; A \in \mathscr{L}(\mathbb{R}^n) \Rightarrow \lambda(T(A)) = |\det T|\lambda(A).$$
- $f : X \to \overline{\mathbb{R}}$ continuous $\Rightarrow f$: measurable w.r.t. $\mathcal{B}(X)$.
- $(X, \mathcal{A})$: measurable space. $f : X \to \overline{\mathbb{R}}$, measurable $\iff \{x \in X : f(x) < c\} \in \mathcal{A}, \forall c \in \mathbb{R}$.
- $(X, \mathcal{A}, \mu)$: complete measure space. $(f_n)_{n\geq 1}$: sequence of measurable functions, $f_n : X \to \overline{\mathbb{R}}$ s.t. $f_n \to f$ pointwise μ-a.e. Then, f is measurable. (Not necessarily continuous.)
- f, g: Lebesgue measurable $\not\Rightarrow f \circ g$: Lebesgue measurable (even if g is continuous).
- f: Borel measurable, g: Lebesgue/Borel measurable $\Rightarrow f \circ g$: Lebesgue/Borel measurable.
- $(X, \mathcal{A}, \mu)$: measure space. $f : X \to [0, \infty]$ measurable. Then, $\exists(\varphi_n)_{n\geq 1}$: sequence of simple functions, pointwise monotone increasing ($\varphi_n(x)$: increasing in $n, \forall x$) and converging pointwise to f.
- $\int_A f d\mu = \int_X f \chi_{X_A} d\mu$.
- **Egoroff's Theorem**:
  $(X, \mathcal{A}, \mu)$: measure space. $\mu(X) < \infty$. $f_n \to f$ a.e., $(f_n)_{n\geq 1}$, f: measurable functions. Then, $\forall \varepsilon > 0$, $\exists B \in \mathcal{A}$ with $\mu(X \setminus B) < \varepsilon$ s.t. $f_n \to f$ uniformly on B.
- Lemma:
  $(X, \mathcal{A}, \mu)$: measure space. $(A_i)_{i\geq 1}$ increasing sequence of measurable sets ($A_i \subset A_{i+1}, \forall i$). Then,
  $$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$$
- Lusin's Theorem:
  $f : [a, b] \to \mathbb{R}$, Lebesgue measurable, $\varepsilon > 0$. Then, $\exists E \subset [a, b]$ compact s.t. $\mu([a, b] \setminus E) < \varepsilon$ and $f|_E$ is continuous.
- Littlewood's 3 principles of measure theory on $\mathbb{R}$:
  1. Every measurable set is nearly a finite sum (union) of intervals.
  2. Every L^p function is nearly continuous.
  3. Every pointwise convergent sequence of functions is nearly uniformly convergent.
- **Monotone Convergence Theorem (MCT)**:
  $(X, \mathcal{A}, \mu)$: measure space. $(f_n)_{n\geq 1}$: a sequence of nonnegative measurable functions s.t. $f_n(x) \leq f_{n+1}(x), \forall x \in X, \forall n \geq 1$. Then,
  $$\int_X \lim_{n\to\infty} f_n(x) d\mu = \lim_{n\to\infty} \int_X f_n(x) d\mu.$$
- **Fatou's Lemma**:
  $(X, \mathcal{A}, \mu)$: measure space. $(f_n)_{n\geq 1}$: sequence of measurable functions, $f_n : X \to \overline{\mathbb{R}}$, $f_n(x) \geq 0, \forall x \in X$. Then,
  $$\int_X \liminf_{n\to\infty} f_n(x) d\mu \leq \liminf_{n\to\infty} \int_X f(x) d\mu.$$

- **Lebesgue's Dominated Convergence Theorem (LDCT)**:
  $(X, \mathcal{A}, \mu)$: measure space. $(f_n)_{n\geq 1}$: sequence of measurable functions $f_n \to f$ a.e. for some measurable f. If $\exists g$ integrable s.t. $|f_n(x)| \leq g(x)$ for a.e. $x \in X$, then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

- Differentiation under integral sign:
  $(X, \mathcal{A}, \mu)$: complete measure space. $I \subset \mathbb{R}$: open. $f : X \times I \to \overline{\mathbb{R}}$, measurable. If
  (i) $f(\cdot, t)$ integrable (on X), $\forall t \in I$,
  (ii) $f(x, \cdot)$ differentiable (in t) for a.e. $x \in X$, and
  (iii) $\exists g : X \to [0, \infty]$ s.t. $\forall t \in I$, $|\partial_t f(x,t)| \leq g(x)$ for a.e. $x \in X$,

  then $t \mapsto \int_X f(x,t)d\mu(x)$ is differentiable in t with derivative

$$\frac{d}{dt}\int_X f(x,t)d\mu(x) = \int_X \frac{\partial}{\partial t} f(x,t)d\mu(x)$$

- Fact:
  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$: σ-finite measure spaces. Then, $\exists!$ measure $\mu\otimes\nu$ on $\mathcal{A}\otimes\mathcal{B}$ s.t. $\forall A \in \mathcal{A}, B \in \mathcal{B}$,

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B).$$

- **Fubini/Tonelli**:
  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ σ-finite measure spaces. $f : X \times Y \to \overline{\mathbb{R}}$, $\mathcal{A} \otimes \mathcal{B}$-measurable. Then, f is $\mu \otimes \nu$-integrable $\iff$
  either

$$\int_X \int_Y |f(x,y)|d\nu(y)d\mu(x) < \infty$$

  or

$$\int_Y \int_X |f(x,y)|d\mu(x)d\nu(y) < \infty$$

  Moreover, if f is $\mu \otimes \nu$-integrable, then

$$\int_{X\times Y} f(x,y)d\mu\otimes\nu = \int_X\int_Y f(x,y)d\nu(y)d\mu(x) = \int_Y\int_X f(x,y)d\mu(x)d\nu(y)$$

- Theorem:
  $(X, \mathcal{A}, \mu)$: measure space. $1 \leq p \leq \infty$. Then, $L^p(X)$ is a Banach space.
- Properties of L^p convergence:
  1. Convergence in L^p does not imply convergence a.e.
  2. $f_n \to f$ in $L^p \Rightarrow f_n \to f$ in measure, i.e., $\forall \varepsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\} \to 0$ as $n \to 0$.
- Proposition:
  $f_n \to f$ in measure $\Rightarrow \exists (f_{n_k})_{k\geq 1}$ s.t. $f_{n_k} \to f$ a.e.