# Microeconomics I

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*Note: This document summarizes contents from the above-named course. The shaded areas denote work I have done on my own and does not pertain to the instructor. All errors are my own.*

## Contents

# 1 Consumer Theory

Framework intellectually fits into the science called Decision Theory (one decision-maker.)
cf. Game Theory: many interacting decision-makers.

Two approaches to basic consumer theory

1. Preferences (classical consumer theory) $\rightsquigarrow$ more unnatural (preference relations), but easier to get to utility.

2. Choices (revealed preference theory)

## 1.1 Classical Consumer Theory

So we begin with classical consumer theory. This model consists of

i) A set of alternatives $X$ (set of options to choose from)

ii) A preference relation on $X$

A preference relation is a binary relation on the set of alternatives.

**Definition 1** (Binary relation)**.** Let $X$ be a nonempty set of alternatives. A binary relation on $X$, denoted by $\mathcal{R}$, is a subset of $X \times X$. If $(x, y) \in \mathcal{R}$, we write $x\mathcal{R}y$, and $x\mathcal{R}y\mathcal{R}z$ means that $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

**Properties of binary relations**

- reflexivity: $\mathcal{R}$ is reflexive if $x\mathcal{R}x, \forall x \in X$.
  ex. indifference ($\sim$) in econ, equality ($=$) in math.

- completeness (totality): $\mathcal{R}$ is complete if either $x\mathcal{R}y$ or $y\mathcal{R}x, \forall x, y \in X$.

- symmetry: $\mathcal{R}$ is symmetric if $x\mathcal{R}y$ then $y\mathcal{R}x$, $\forall x, y \in X$.

- asymmetric: $\mathcal{R}$ is asymmetric if $x\mathcal{R}y$ then $-y\mathcal{R}x$, $\forall x, y \in X$.
  ex. "$>$" on $\mathbb{R}$.

- transitivity: $x\mathcal{R}y$ and $y\mathcal{R}z \Rightarrow x\mathcal{R}z, \forall x, y, z \in X$.
  ex. "$\geq$" on $\mathbb{R}$

- antisymmetry: $x\mathcal{R}y\mathcal{R}x \Rightarrow x = y, \forall x, y \in X$.

- $\mathcal{R}$ is a preorder if it is reflexive and transitive.

- $\mathcal{R} \subseteq X \times X$ is a partial order if and only if it is reflexive, transitive and antisymmetric.

- $\mathcal{R}$ is a linear order if it is a complete partial order.

Properties of the consumption set $X$

1. $X \subseteq \mathbb{R}^n_+$
2. $X$ is closed
3. $X$ is convex
4. $\mathbf{0} \in X$

### 1.1.1 Preference

A consumer's preference relation will be described as a binary relation on $X$ and we denote it by "$\succsim$". ("weakly preferred to")

- $x \succsim x'$ means $x$ is weakly preferred to $x'$.
- $\succsim$ is ordinal in nature (not cardinal) and hence says NOTHING about INTENSITY

A preference relation is *rational* if it satisfies the following axioms:

Axiom 1. (Completeness) $\forall x^1, x^2 \in X$ either $x^1 \succsim x^2$ or $x^2 \succsim x^1$.

Axiom 2. (Transitivity) $\forall x^1, x^2, x^3 \in X$, $x^1 \succsim x^2$ and $x^2 \succsim x^3 \Rightarrow x^1 \succsim x^3$.

**Theorem 1.** *Given a finite set of alternatives $A \subseteq \mathbb{R}^n_+$, if $\succsim$ on $A$ is rational, then elements in $A$ can be ranked in a way consistent with $\succsim$.*

*Proof.* I provide a proof by mathematical induction on $n$. Let $|A| = n$.

i) (Base case) $n = 1$.
We have $A = \{a\}$, which is already ranked.

ii) (Induction step) $n = k$.
Suppose for $n = k$, we can rank elements in a set consistent with $\succsim$. Consider a set $A$ with $k + 1$ elements. Then, define $A' := A \setminus \{a\}$, where $a$ is any element in $A$.

Now, $A'$ is a set with cardinality $k$. By the inductive hypothesis, we can rank it consistently with $\succsim$, i.e., $A' = \{a_1, \ldots, a_k\}$ and $a_1 \succsim a_2 \succsim \cdots \succsim a_k$.

If $a \succsim a_1$, then we give it the rank of one and add one to the rank of all others as it will imply $a \succsim a_i, \forall i \in \{1, \ldots, k\}$ by transitivity. If not, then we must have $a_1 \succsim a$ by completeness. We then take $a_2$, and if $a \succsim a_2$, we rank it as second and add one to the rank of elements from $a_2$, as it will imply $a \succsim a_i, \forall i \in \{2, \ldots, k\}$ by transitivity. If not, we must have $a_2 \succsim a'$ by completeness.

Continue on in this fashion until we find $a_{k'} \succsim a \succsim a_{k'+1}$ and assign the rank $k'+1$ and push all other elements back by 1. If we cannot find such $k'$, then we must have $a_k \succsim a$, in which case we assign $k+1$.

Hence, with this algorithm, we have ranked a set $A$ with cardinality $k+1$.

Hence, by induction, we can rank a finite set $A$ in a way consistent with a rational $\succsim$ on $A$. □

The ranking provides all the information in $\succsim$, which can be put into a function, namely *utility*. In fact, utility is just a function to facilitate discussions about preference.

**Definition 2** (Indifference)**.** The indifference relation "$\sim$" is defined by $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$.

**Definition 3** (Strict preference)**.** The strict preference relation, denoted by $\succ$, is defined by $x \succ$ iff $x \succsim y$ but not $y \succsim x$.

*Remark.* $x \succ y$, $x \sim y$, $y \succ x$ are mutually exclusive.

Some sets

1. At-least-as-good set: $\succsim (x^0) := \{x \in \mathbb{R}^n_+ : x \succsim x^0\}$
2. No-better-than set: $(x^0) \succsim := \{x \in \mathbb{R}^n_+ : x^0 \succsim x\}$
3. Strictly-preferred-to set: $\succ (x^0) := \{x \in \mathbb{R}^n_+ : x \succ x^0\}$
4. Strictly-worse set: $(x^0) \succ := \{x \in \mathbb{R}^n_+ : x^0 \succ x\}$
5. Indifference set: $\sim (x^0) := \{x \in \mathbb{R}^n_+ : x^0 \sim x\}$

*Remark.* $\sim (x^0) = \succsim (x^0) \cap (x^0) \succsim = \mathbb{R}^n_+ \setminus (\succ (x^0) \cup (x^0) \succ)$.

Axiom 3. (Continuity) $\succsim$ is said to be continuous if for any $x^0 \in \mathbb{R}^n_+$, the sets $\succsim (x^0)$ and $(x^0) \succsim$ are closed in $\mathbb{R}^n_+$.

Axiom 4'. (Local nonsatiation) $\forall x^0 \in \mathbb{R}^n_+, \forall \varepsilon > 0$, $B_\varepsilon(x^0) \cap \mathbb{R}^n_+$ contains a bundle in $\succ (x^0)$.

*Remark.* Local nonsatiation rules out bliss points.

Axiom 4. (Strict monotonicity) $\forall x^0, x^1 \in \mathbb{R}^n_+$, if $x^0 \geq x^1$ then $x^0 \succsim x^1$ and if $x^0 \gg x^1$ then $x^0 \succ x^1$.

Axiom 5'. (Convexity) If $x^1 \succsim x^0$, then $tx^1 + (1-t)x^0 \succsim x^0, \forall t \in [0,1]$.

Axiom 5. (Strict convexity) If $x^1 \succsim x^0$ and $x^1 \neq x^0$, then $tx^1 + (1-t)x^0 \succ x^0, \forall t \in (0,1)$.

### 1.1.2 Utility

**Definition 4** (Utility function)**.** A function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called a utility function representing "$\succsim$" if for all $x^1, x^2 \in \mathbb{R}^n_+$, $u(x^1) \geq u(x^2) \iff x^1 \succsim x^2$.

*Remark.* Note that, if $\succsim$ violated transitivity, we cannot represent it with a utility function.

**Theorem 2** (Existence of a continuous real-valued function representing $\succsim$)**.** *If the binary relation $\succsim$ is complete, transitive, continuous and strictly monotonic, there exists a continuous real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents "$\succsim$".*

*Proof.* Let $\mathbb{e} = \mathbb{1} \in \mathbb{R}^n_+$. For any $x \in \mathbb{R}^n_+$, define $u(x)$ so that $x \sim u(x) \cdot \mathbb{e}$. We would like to show that such $u(x)$ exists and that it is unique.

Consider the following two sets:

$$A := \{t \geq 0 : t \cdot \mathbb{e} \succsim x\}, B := \{t \geq 0 : x \succsim t \cdot \mathbb{e}\}.$$

Note that if $t^* \in A \cap B$, then $u(x) := t^*$ and we are done. It now suffices to show $A \cap B \neq \phi$. First, note that $A \cap B$ is closed, since $A, B$ are closed by continuity of $\succsim$. Moreover, strict monotonicity and transitivity imply that, if $t_0 \in A$ (i.e., $t_0 \cdot \mathbb{e} \succsim x$), then $t \cdot \mathbb{e} \succsim t_0 \cdot \mathbb{e}, \forall t \geq t_0$ $\Rightarrow t \cdot \mathbb{e} \succsim x$.

Because of this, we know that $t \in A$, $A = [\underline{t}, \infty)$ for some $\underline{t}$. Similarly, $B = [0, \bar{t}]$ ($\because$ If $t' \in B$, that means $t'\mathbb{e} \precsim x$, then for all $t'' < t', t'' \cdot \mathbb{e} \precsim t' \cdot \mathbb{e} \Rightarrow t'' \cdot \mathbb{e} \precsim x \Rightarrow t'' \in B$.)

Consider $t \geq 0$, and take $x \in \mathbb{R}^n_+$. Now, by completeness, we know that either $t \cdot \mathbb{e} \succsim x$ or $t \cdot \mathbb{e} \precsim x$, so $A \cup B = \mathbb{R}_+$. Therefore, $A \cap B$ cannot be empty.

Now we prove uniqueness. Suppose $\underline{t} < \tilde{t} < \bar{t}$ such that $\tilde{t} \in A$ and $\tilde{t} \in B$. Together, this implies $\tilde{t}\mathbb{e} \sim x$. Consider $\tilde{t} < \hat{t}$ such that $\hat{t} \in A$ and $\hat{t} \in B \Rightarrow \hat{t}\mathbb{e} \sim x \Rightarrow \tilde{t}\mathbb{e} \sim \hat{t}\mathbb{e}$ but this violates strict monotonicity since $\hat{t}\mathbb{e} \gg \tilde{t}\mathbb{e}$. This is a contradiction. Therefore, $\underline{t} < \bar{t}$ cannot be true. We also know $A \cup B = \mathbb{R}_+$, so we must have $\underline{t} \geq \bar{t}$. Hence, $\underline{t} = \bar{t}$.

For each $x$, assign $t^* \in A \cap B$ as $u(x)$. Since $u(x)$ exists for any $x \in \mathbb{R}^n_+$ and is unique, $u(\cdot)$ is indeed a function.

I now show that $u(\cdot)$ is a utility function that represents $\succsim$.

($\Rightarrow$) Suppose $x^1 \succsim x^2$. By definition, $u(x^1)\mathbb{e} \sim x^1$ and $u(x^2)\mathbb{e} \sim x^2$. Then, by transitivity of $\succsim$, we have $u(x^1)\mathbb{e} \succsim u(x^2)\mathbb{e}$. By strict monotonicity, we can conclude that $u(x^1) \geq u(x^2)$ as desired.

($\Leftarrow$) Suppose $u(x^1) \geq u(x^2)$. By strict monotonicity, $u(x^1)\mathbb{e} \succsim u(x^2)\mathbb{e}$. By definition, $u(x^1)\mathbb{e} \sim x^1$ and $u(x^2)\mathbb{e} \sim x^2$. By transitivity, we have $x^1 \succsim x^2$ as desired.

Finally, it remains to show that $u : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous. It suffices to show that $u^{-1}((a,b)) = \{x \in \mathbb{R}^n_+ : a < u(x) < b\}$ is open in $\mathbb{R}^n_+$, $\forall a, b \in \mathbb{R}$.

$$\Rightarrow u^{-1}((a,b)) = \{x \in \mathbb{R}^n_+ : a\mathbb{e} \prec u(x)\mathbb{e} \prec b\mathbb{e}\} = \succ(a\mathbb{e}) \cap (b\mathbb{e})\succ .$$

Note that $\succ(a\mathbb{e}) = ((a\mathbb{e})\succsim)^c$ and $(b\mathbb{e})\succ = (\succsim(b\mathbb{e}))^c$. By continuity of $\succsim$, $(a\mathbb{e})\succsim$ and $\succsim(b\mathbb{e})$ are closed, and thus their complements are open. Hence, $u^{-1}((a,b))$ is a finite intersection of open sets and thus itself open as desired. Since $(a,b)$ was an arbitrary open interval, $u(\cdot)$ is a continuous function. □

Utility functions are NOT unique. Always keep in mind that numbers have ordinal meaning.

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**Example 1.** If $u : \mathbb{R}^n_+ \to \mathbb{R}$ represents $\succsim$ and $v : \mathbb{R}^n_+ \to \mathbb{R}$ also represents $\succsim$ then $v(x) = g(u(x)), \forall x \in \mathbb{R}^n_+$ for some strictly increasing $g : \mathbb{R} \to \mathbb{R}$.

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Examples of utility functions in $\mathbb{R}^2_+$

1. Perfect complements (Leontief preferences)
   $u(x_1, x_2) = \min\{x_1, x_2\}$
2. Perfect substitutes
   $u(x_1, x_2) = x_1 + x_2$
3. Cobb-Douglas utility
   $u(x_1, x_2) = x_1^\alpha x_2^\beta$ (has nice properties, so used often)

Properties of preference relation translate to properties of the indifference set and these turn into properties of the utility function. Notably, the axiom of convexity has implications on the curvature of indifference sets. Convexity implies that the absolute value of the slope of indifference curves is decreasing.

**Theorem 3.** *Let $\succsim$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then,*

1. *$u(\cdot)$ is strictly increasing if and only if $\succsim$ is strictly monotonic.*
2. *$u(\cdot)$ is quasiconcave[1] if and only if $\succsim$ is convex.*
3. *$u(\cdot)$ is strictly quasiconcave if and only if $\succsim$ is strictly convex.*

### 1.1.3 Marshallian Demand

Solving the consumer's problem: The consumer has a preference relation that can be represented by a continuous utility function.

1. Quasiconcavity: $u(tx^0 + (1-t)x^1) \geq \min\{u(x^0), u(x^1)\}$.

A budget set $B = \{\text{all the bundles the consumer can choose from}\}$. $x^* \in B : x^* \succsim x, \forall x \in B \rightsquigarrow$ most preferred in the budget set (optimizes)

The following formulation incorporates two important assumptions:

1. optimizing behavior (the consumer chooses the best among available options)
2. existence of a competitive market for each good, which means that the consumer is a price-taker: faces a vector of prices that are fixed.

$B(\mathbb{p}, I) = \{x \in \mathbb{R}^n_+ : \mathbb{p} \cdot x \leq I\}$ where $\mathbb{p} \in \mathbb{R}^n_+$ is the price vector and $I \in \mathbb{R}_+$ is the consumer's income.

Assuming $\succsim$ is represented by $u(\cdot)$, we write $x^* \in B$ s.t. $x^* \succsim x, \forall x \in B \iff$

$$\max_{x \in B(p,I)} u(x), \quad x^* = \arg\max_{x \in B(p,I)} u(x)$$

When $n = 2$, $B(p, I) = \{(x_1, x_2) \in \mathbb{R}^n_+ : p_1 x_1 + p_2 x_2 \leq I\}$. $p_1 x_1 + p_2 x_2 = I$ (boundary of the budget set.

We want to show that the maximization problem has a solution. This would be done using Weierstrass theorem, because $u(\cdot)$ is continuous and $B(p, I)$ is compact. To show the latter, it suffices to show that $B(p, I)$ is closed and bounded (by Heine-Borel theorem).

i) $B(p, I)$ is bounded:
$0 = (0, 0, \ldots, 0)$ is a lower bound, $\left(\frac{I}{p_1}, \ldots, \frac{I}{p_n}\right)$ is an upper bound.

ii) $B(p, I)$ is closed:

Just to reiterate: $B(p, I) = \{(x_1, x_2) \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq I\}$.

Consider a sequence $(x_{1,n}, x_{2,n})_{n=1}^{\infty} \subseteq B(p, I)$ such that $(x_{1,n}, x_{2,n}) \to (x_1^*, x_2^*) \in \mathbb{R}^2$. Then, $p_1 x_{1,n} + p_2 x_{2,n} \leq I, \forall n$. Define $f(x_1, x_2) = p_1 x_1 + p_2 x_2$. Note that $f$ is a continuous function on $\mathbb{R}^2$.[a] Then, we have $y_n := f(x_{1,n}, x_{2,n}) \to f(x_1^*, x_2^*) =: y^*$. Note that $y_n$ is a sequence in $[0, I] \subset \mathbb{R}$, which is a closed set. Therefore, $y^* = f(x_1^*, x_2^*) \in [0, I]$, which implies $B(p, I)$ is closed.

*a.* Let $(x_1, x_2) \in \mathbb{R}^2$, $\varepsilon > 0$ be given. Take $\delta := \frac{\varepsilon}{p_1 + p_2}$.
Consider $(x_1', x_2') \in \mathbb{R}^2$ with $\|(x_1', x_2') - (x_1, x_2)\| < \delta$. Note that, we have $|x_1' - x_1| \leq \sqrt{(x_1' - x_1)^2 + x_2' - x_2)^2} < \delta$. Then,

$$\begin{aligned} |f(x_1', x_2') - f(x_1, x_2)| &= |p_1 x_1' + p_2 x_2' - p_1 x_1 - p_2 x_2| \\ &\leq |p_1 x_1' - p_1 x_1| + |p_2 x_2' - p_2 x_2| \\ &< p_1 \delta + p_2 \delta < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0, (x_1, x_2)$ were arbitrary, $f$ is continuous.

We would like the solution to be unique as well (as it is easier to handle). Otherwise we would have a demand correspondence instead of a demand function.

To ensure uniqueness of consumer's optimal bundle, we need strict convexity of the preference relation, which implies strict quasiconcavity of the utility representation.

We will show that, if $u(\cdot)$ is strictly quasiconcave, then we have only one solution:

$$x(p, I) = (x_1(p, I), \ldots, x_n(p, I))$$

and the solution is then called the *Marshallian demand function*.

Now we argue the uniqueness under quasiconcavity. For a contradiction, suppose that there are two solutions, say $x^1$ and $x^2$. Because $x^1, x^2$ are solutions, they are feasible, i.e., $p \cdot x^1 \leq I$ and $p \cdot x^2 \leq I$. For some $t \in [0, 1]$, $p \cdot tx^1 \leq tI$, $p \cdot (1-t)x^2 \leq (1-t)I$.

$$\Rightarrow p \cdot (tx^1 + (1-t)x^2) \leq tI + (1-t)I = I.$$

Let $\bar{x} = tx^1 + (1-t)x^2$. Then, $\bar{x}$ is also feasible, but $u(\cdot)$ is strictly quasiconcave, i.e.,

$$u(\bar{x}) > \min\{u(x^1), u(x^2)\}.$$

This strict inequality contradicts the optimality of $x^1, x^2$, which is a contradiction. Therefore, we must have a unique solution.

When $u(\cdot)$ is strictly increasing, the constraint binds. In other words, we can impose it as an equality ($p \cdot x = I$). Suppose $x^*$ solves the problem and $p \cdot x^* < I$. Then, $x^* + \varepsilon$ with $p(x^* + \varepsilon) \leq I$ for $\varepsilon > 0$ small enough. Then, $u(x^* + \varepsilon) > u(x^*)$, which is a contradiction to the fact that $x^*$ is a maximizer.

From now on, we assume that $u(\cdot)$ is differentiable and we rely on Lagrangian methods.

$$\mathcal{L}(x, \lambda) = u(x) + \lambda(I - p \cdot x).$$

To get to a solution, we differentiate the Lagrangian with respect to $x, \lambda$.

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial x_i} = \frac{\partial u(x^*)}{\partial x_i} - \lambda^* p_i = 0, \forall i \in \{1, \ldots, n\}.$$

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial \lambda} = I - p \cdot x^* = 0$$

These first-order necessary conditions assume an interior solution $x^* \gg 0$. (This is not always the case, as we can have corner/boundary solutions for perfect substitutes).

We also assume:

$$\frac{\partial u(x_1, x_2)}{\partial x_i} = \text{MU}_i > 0$$

where $\text{MU}_i$ is the marginal utility of good $i$.

$$\frac{\partial u(x_1, x_2)}{\partial x_i} = \lambda^* p_i, \quad \frac{\partial u(x_1, x_2)}{\partial x_j} = \lambda^* p_j, \text{ for any two goods } i, j.$$

Note that, in the case of $n = 2$,

$$\frac{\partial u(x_1, x_2)}{\partial x_1} = \lambda p_1, \frac{\partial u(x_1, x_2)}{\partial x_2} = \lambda p_2 \Rightarrow \frac{\text{MU}_1}{\text{MU}_2} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}.$$

The LHS is the slope of the indifference curve and the RHS is the slope of the budget line.

Note that, along an indifference curve, $dU = 0$. Hence, we can write

$$dU = \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0$$

$$\Rightarrow \frac{dx_1}{dx_2} = -\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = -\frac{\text{MU}_1}{\text{MU}_2}$$

For any bundle $x \in \mathbb{R}^n_+$, we can calculate the marginal utilities of <u>all</u> goods at that bundle.

$$\nabla U = \left( \frac{\partial u(x)}{\partial x_1}, \ldots, \frac{\partial u(x)}{\partial x_n} \right),$$

which is the gradient.

- Price vector $(p_1, p_2)$ is orthogonal to the budget line.
- Take a bundle $x^0$; the gradient of the utility $u(\cdot)$ at $x^0$ is also orthogonal to the indifference curve through $x^0$.

$$\left( \frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_n} \right) \cdot (dx_1, \ldots, dx_n) = \frac{\partial U}{\partial x_1} dx_1 + \cdots + \frac{\partial U}{\partial x_n} dx_n = 0$$

Because, at an optimal bundle, the slope of the budget line and the indifference curve are the same, this means that the gradient vector $\nabla U$ is proportional to the price vector, i.e., there exists $\lambda^*$ such that

$$\nabla u(x^*) = \lambda^* p$$

$$\left(\frac{\partial u(x^*)}{\partial x_1}, \ldots, \frac{\partial u(x^*)}{\partial x_n}\right) = (\lambda^* p_1, \ldots, \lambda^* p_n)$$

by Kuhn-Tucker theorem (to be discussed).

**Theorem 4.** *Suppose $u(\cdot)$ is strictly quasiconcave and differentiable on $\mathbb{R}^n_{++}$ and $(p, I) \gg 0$. Then, if $(x^*, \lambda^*) \gg 0$ solve the FOCs, then $x^*$ solves the consumer's problem.*

*Proof.* The proof uses the following property of strictly quasiconcave and differentiable functions. The property is that if $x \neq x'$ and $\nabla u(x) \neq 0$ and $u(x') > u(x)$ then

$$\nabla u(x)(x' - x) > 0 \quad (*)$$

We use $(*)$ to show that if $x^*$ solves the FOC then it is a solution. For a contradiction, suppose $x^*$ is not a utility maximizing choice. Then, $\exists x'$ with $p \cdot x \leq I$ s.t. $u(x') > u(x^*)$. We know that $\nabla u(x^*) \neq 0$ ($\because$ $\nabla u(x^*) = \lambda^* p$, where $\lambda^* > 0$ and $p \gg 0 \Rightarrow \nabla u(x^*) \gg 0$). Then, by $(*)$, $\nabla u(x^*)(x' - x^*) > 0$

$$\begin{aligned}
&\Rightarrow \lambda^* p \cdot (x' - x^*) > 0 \;\; (\because \nabla u(x^*) = \lambda^* p) \\
&\Rightarrow \lambda^* (p \cdot x' - p \cdot x^*) > 0 \;\; (\because \text{ associativity}) \\
&\Rightarrow p \cdot x' > p \cdot x^* \;\; (\because \lambda^* > 0) \\
&\Rightarrow p x' > I,
\end{aligned}$$

which is a contradiction. □

Consumer's Problem:

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ s. t. } x \in B(p, I)$$

$$x^*(p, I) = (x_1^*(p, I), \ldots, x_n^*(p, I)).$$

Define the indirect utility function $V$ as

$$u(x^*(p, I)) = V(p, I).$$

This is the value function of the consumer's problem. The maximum utility the consumer can achieve as a function of prices and $I$. (Feeds solutions of maximization into utility.)

**Theorem 5** (Properties of the indirect utility function.)**.** *If $u(\cdot)$ is strictly increasing and continuous on $\mathbb{R}^n_+$, then $V(p, I)$ is*

*i) continuous on its domain $\mathbb{R}^n_{++} \times \mathbb{R}_+$*

*ii) homogeneous of degree zero in prices and income*

iii) *strictly increasing in $I$*

iv) *decrasing in $p$*

v) *quasiconvex in in $(p, I)$*

vi) *if $V(p^0, I^0)$ is differentiable and $\frac{\partial V}{\partial I} \neq 0$, then it satisfies Roy's identity:*

$$x_i(p^0, I^0) = -\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial I}}$$

*i.e., we can find the demand from two derivatives of $V$.*

*Proof.*

i) The fact that $V(p, I)$ is continuous follows from Berge's theorem of the maximum. This is because $u(\cdot)$ is continuous and the constraint set varies continuously with $(p, I)$.

ii) Homogeneity of degree zero
For $k > 0$,

$$B(kp, kI) = \{x : kp \cdot x \leq kI\} = \{x : p \cdot x \leq I\} = B(p, I).$$

The budget set and the utility function are the same, so the solutions to maximization are the same.

iii) Strictly increasing in $I$
Suppose $x(p, I)$ is the consumer's demand at prices $p$ and income $I$. Consider $I' > I$. Then, $p \cdot x(p, I) \leq I < I'$. For some $\varepsilon > 0$, $p \cdot (x(p, I) + (\varepsilon, \dots, \varepsilon)) \leq I'$. Call this bundle $\tilde{x}$.

$$V(p, I') \geq u(\tilde{x}) \ \ (\because V(p, I') = u(x^*(p, I')) \text{ and } x^* \text{ is the maximizer.})$$

$$\Rightarrow V(p, I') \geq u(\tilde{x}) > u(x(p, I)) = V(p, I) \Rightarrow V(p, I') > V(p, I)$$

as desired.

iv) Decreasing in $p$

Suppose $x(p, I)$ is the consumer's demand at prices $p$ and income $I$. Consider $p' \gg p$. Then, we have $p \cdot x(p', I) \leq p' \cdot x(p', I) \leq I$ as $p, p' \gg 0, x \geq 0$. Note that this implies $x(p', I)$ is feasible for the following problem:

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ subject to } p \cdot x \leq I.$$

Then, we must have

$$u(x(p,I)) \geq u(x(p',I))$$

as $x(p,I)$ is the maximizer. Hence,

$$V(p,I) \geq V(p',I)$$

as desired.

v) Quasiconvexity in $(p,I)$

$(p^0,I^0),(p',I'),t \in [0,1]$, $t(p^0,I^0)+(1-t)(p',I') = (tp^0+(1-t)p', tI^0+(1-t)I')$. Let $\bar{p} := tp^0+(1-t)p'$ and $\bar{I} = tI^0+(1-t)I'$. Then,

$$V(tp^0+(1-t)p', tI^0+(1-t)I') \leq \max\{V(p^0,I^0), V(p',I')\}.$$

We show that $B(p^0,I^0) \cup B(p',I') \supseteq B(\bar{p},\bar{I})$, i.e., $x \in B(\bar{p},\bar{I})$, then either $x \in B(p^0,I^0)$ or $x \in B(p',I')$.

For a contradiction, suppose there exists $\bar{x} \in B(\bar{p},\bar{I})$ s.t. $\bar{x} \notin B(p^0,I^0)$ and $\bar{x} \notin B(p',I')$. This implies $p^0 \cdot \bar{x} > I^0, p' \cdot \bar{x} > I'$. Then,

$$\begin{aligned}
&tp^0 \cdot \bar{x} > tI^0, (1-t)p' \cdot \bar{x} > (1-t)I' \\
\Rightarrow &(tp^0+(1-t)p') \cdot \bar{x} > tI^0+(1-t)I' \\
\Rightarrow &\bar{p}\bar{x} > \bar{I},
\end{aligned}$$

which is a contradiction.

vi) Roy's identity

Version I:

Using envelope theorem.[2]

Note that the Lagrangian for the maximization problem is

$$\mathcal{L}(p,I) = u(x^*(p,I)) - \lambda^*(p,I)(p \cdot x^*(p,I) - I).$$

2. The envelope theorem: By chain rule, we have

$$f(p, x^*(p)) \rightsquigarrow \frac{df}{dp} = \frac{\partial f}{\partial p} + \frac{\partial f}{\partial x^*}\frac{\partial x^*}{\partial p}.$$

At optimum, we will have $\frac{\partial x^*}{\partial p} = 0$. Therefore, it suffices to consider $\frac{df}{dp} = \frac{\partial f}{\partial p}$.

Since the constraint will bind at the optimum given strictly increasing $u(\cdot)$, we have

$$\mathcal{L}(p, I) = u(x^*(p, I)) = V(p, I).$$

Now, by the envelope theorem, we have

$$\frac{\partial \mathcal{L}}{\partial I} = \lambda^*(p, I)$$

and

$$\frac{\partial \mathcal{L}}{\partial p_i} = -\lambda^*(p, I)x_i.$$

Rearranging the second equation and substituting the first gives

$$x_i = -\frac{\frac{\partial \mathcal{L}(p,I)}{\partial p_i}}{\lambda^*(p, I)} = -\frac{\frac{\partial \mathcal{L}(p,I)}{\partial p_i}}{\frac{\partial \mathcal{L}(p,I)}{\partial I}} = -\frac{\frac{\partial V(p,I)}{\partial p_i}}{\frac{\partial V(p,I)}{\partial I}}$$

as desired.

Version II:
Let $x^0 = x(p^0, I^0)$. Consider the function $V(p, p \cdot x^0) - u(x^0)$. This is a function of $p \in \mathbb{R}^n_{++}$. Notice that $x^0$ is affordable for the problem:

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ s.t. } p \cdot x \leq p \cdot x^0.$$

Then,

$$G(p) := V(p, p \cdot x^0) - u(x^0) \geq 0, \forall p \in \mathbb{R}^n_{++},$$

since $p \cdot x^0 \leq p \cdot x^0$ (i.e., $x^0$ is affordable) and $V$ is the maximum. The function is positive for all $p$ and 0 at $p^0$, and hence it must be minimized at $p^0$. This implies that the gradient of $G$ around $p^0$ must be equal to

$$\frac{\partial G(p)}{\partial p_i} = \frac{\partial V(p^0, p^0 \cdot x^0)}{\partial p_i} + \frac{\partial V(p^0, p^0 \cdot x^0)}{\partial I} x_i^0 = 0, \forall i = 1, \ldots, n.$$

Rearranging gives

$$x_i^0(p^0, I^0) = -\frac{\frac{\partial V(p^0,I^0)}{\partial p_i}}{\frac{\partial V(p^0,I^0)}{\partial I}}.$$

□

**Example 2.** Derive the Marshallian demand for a CES (constant elasticity of substitution)

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, \rho \neq 0, \rho < 1.$$

The Lagrangian for the maximization problem is:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1}\rho x_1^{\rho-1} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1}\rho x_2^{\rho-1} - \lambda p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_1 x_1 - p_2 x_2 = 0$$

After some algebra,

$$x_2^*(p_1, p_2, \lambda) = \frac{I p_2^{-\frac{1}{1-\rho}}}{p_1^{-\frac{\rho}{1-\rho}} + p_2^{-\frac{\rho}{1-\rho}}}$$

and $x_1^*$ is analogous.

Let $r = -\frac{\rho}{1-\rho}$. By substituting $x_1^*, x_2^*$ into $u(\cdot)$, we obtain the indirect utility function

$$V(p_1, p_2, I) = I(p_1^r + p_2^r)^{-\frac{1}{r}}.$$

From Roy's identity, we can get back say $x_1^*$:

$$\frac{\partial V}{\partial I} = (p_1^r + p_2^r)^{-\frac{1}{r}}, \quad \frac{\partial V}{\partial p_1} = -\frac{I}{r}(p_1^r + p_2^r)^{-\frac{1}{r}-1} r p_1^{r-1}$$

$$\Rightarrow x_1^*(p, I) = -\frac{\frac{\partial V}{\partial p_1}}{\frac{\partial V}{\partial I}} = \frac{I(p_1^r + p_2^r)^{-\frac{1}{r}-1} p_1^{r-1}}{(p_1^r + p_2^r)^{-\frac{1}{r}}} = \frac{I p_1^{r-1}}{p_1^r + p_2^r}.$$

---

#### 1.1.4 Hicksian Demand

Now, consider a different problem: given a utility level, what is the cheapest way to get to it?

The consumer's expenditure minimization problem. Suppose prices are fixed at some level $p$ and the consumer seeks to achieve a given utility level $u$. What is the cheapest way to do so? In other words, what is the cheapest bundle that achieves utility $u$ at prices $p$?

Formally,

$$\min_{x \in \mathbb{R}_+^n} p \cdot x \text{ s.t. } u(x) \geq u$$

This is the expenditure minimization problem and mathematically is the dual of the utility maximization problem. The value function is denoted by $e(p, u)$ and it is called the *expenditure function*. Solutions are called *Hicksian demand*, denoted $x^h(p, u)$:

$$e(p, u) = p \cdot x^h = \sum_{i=1}^{n} p_i x_i^h(p, u).$$

This may be an unnatural problem to think about. However, Hicksian demand has nice properties, such as always being downward sloping (whereas Marshallian demand may not). In order to get to more sophisticated results, it is useful to think about demand in terms of the Hicksian demand.

First, we need to investigate whether the problem is well defined. Does it have a solution?

Let $U := \{u \in \mathbb{R} : u(x) = u, x \in \mathbb{R}^n_+\}$. If $u \in U$, by definition, $\exists x^0 \in \mathbb{R}^n_+$ s.t. $u(x^0) = u$.[3] Then, $p \cdot x^0$ is enough money to achieve $u$. Therefore,

$$e(p, u) \leq p \cdot x^0 \quad (\because e(\cdot) \text{ is the minimum.})$$

Now, let us look at the following formulation of the problem:

$$\min_{x \in \mathbb{R}^n_+} p \cdot x \text{ s.t. } u(x) \geq u, p \cdot x \leq p \cdot x^0.$$

The solution is the same because the solution will trivially satisfy the second constraint. However, now we are minimizing a continuous function over a compact set. Therefore, a solution exists.

Next, we give conditions that guarantee a unique solution:
We show that if $u(\cdot)$ is continuous, strictly increasing, strictly quasiconcave and $p \gg 0$, then the solution is unique.

For a contradiction, let $x^0, x^1$ be two different solutions. Then, $u(x^0) \geq u, u(x^1) \geq u$. Strict quasiconcavity of $u(\cdot)$ implies:

$$u\left(\frac{1}{2}x^0 + \frac{1}{2}x^1\right) > \min\{u(x^0), u(x^1)\} \geq u.$$

Note also that $p \cdot x^0 = p \cdot x^1$ because both $x^0, x^1$ are solutions. This implies that for $\bar{x} = \frac{1}{2}x^0 + \frac{1}{2}x^1$,

$$p \cdot \bar{x} = p \cdot \left(\frac{1}{2}x^0 + \frac{1}{2}x^1\right) = p \cdot x^0 = p \cdot x^1.$$

Reduce $\bar{x}$ by $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Then, $\bar{x} - \varepsilon$ satisfies $u(\bar{x} - \varepsilon) \geq u$ for $\varepsilon$ small enough and

$$p \cdot (\bar{x} - \varepsilon) < p \cdot \bar{x} = p \cdot x^0 = p \cdot x^1.$$

This is a contradiction to the fact that $x^0, x^1$ are minimum expenditure solutions. Hence, we must have a unique solution.

Let $x^h(p, u)$ be the unique solution, which is called the *Hicksian demand*. Suppose $u(\cdot)$ is differen-

3. Some caveats exist in drawing this result. The level $u$ has to be actually achievable, i.e., be in the image of $u(\cdot)$. I guess there are various ways to motivate this.

tiable. Then, we can solve the problem using Lagrangian:

$$\mathcal{L}(x, \lambda) = p \cdot x - \lambda(u(x) - u).$$

Assuming an interior solution $(x^*, \lambda^*)$, Lagrange's theorem tell us that the solution satisfies the following conditions:

$$\frac{\partial \mathcal{L}}{\partial x_i}(x^*, \lambda^*) = p_i - \lambda^* \frac{\partial u(x^*)}{\partial x_i} = 0, \forall i = 1, \dots, n \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x^*, \lambda^*) = u(x^*) - u = 0 \tag{2}$$

Solve this system to get $x^*(p, u), \lambda^*(p, u)$. If we replace the solution to the objective function:

$$\mathcal{L} = p \cdot x^*(p, u) - \lambda^*(p, u)(\underbrace{u(x^*(p, u)) - u}_{=0, \text{ by } (2)})$$

$$\Rightarrow \mathcal{L}(x^*, \lambda^*) = p \cdot x^*(p, u) \equiv e(p, u)$$

The function $e : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is the value function of the expenditure minimization problem and we call it the *expenditure function.* (This whole process is identical to cost minimization problem for the producer.)

**Theorem 6** (Properties of the Expenditure Function)**.** *Suppose $u : \mathbb{R}^n_+ \to \mathbb{R}$ is strictly increasing and continuous. Then, $e(p, u)$*

1. *is zero when $u(\cdot)$ takes its lowest possible level $u$;*
2. *is continuous on $\mathbb{R}^n_{++} \times \mathbb{R}$;*
3. *is, for $p \gg 0$, strictly increasing and unbounded in $u \in U$;*
4. *is nondecreasing in $p$, holding $u$ fixed;*
5. *is homogeneous of degree 1 in $p$;*
6. *is concave in $p$; and*
7. *satisfies the Shephard's lemma at $(p^0, u^0)$ with $p^0 \gg 0$, i.e.,*

$$\frac{\partial e(p^0, u^0)}{\partial p_i} = x^h_i(p^0, u^0).$$

*Proof.*

1. $u(\cdot)$ is minimized at $x = (0, \dots, 0)$ because its domain is $\mathbb{R}^n_+$ and it is strictly increasing so $\underline{u} = u(0)$ and $p \cdot 0 = 0$ for any $p$. Hence, $e(p, \underline{u}) = 0$.

2. Continuity follows from the theorem of the maximum, given that the objective function is continuous in its arguments and the feasible set is compact.

3. i) Increasing in $u$:
   Suppose $u' > u$. We would like to show that $e(p, u') > e(p, u)$.
   Let $p \cdot x' = e(p, u'), u(x') \geq u$ and $p \cdot x = e(p, u), u(x) \geq u$. Since the constraints will bind, we have
   $$u(x') = u', \;\; u(x) = u \Rightarrow u(x') > u(x).$$
   Now, we have $p \cdot x = e(p, u) \leq p \cdot x' = e(p, u')$. To get a strict inequality, it suffices to show that $p \cdot x \neq p \cdot x'$.
   For a contradiction, suppose $p \cdot x = p \cdot x'$. Since $u(x') \geq u' > u \geq u(0)$[4], $x' \neq 0$ since $u$ is a strictly increasing function. Then, there exists a good $i$ with $x'_i > 0$. For $\varepsilon > 0$ small enough, we have
   $$u(\tilde{x}) = u(x'_1, \ldots, x'_i - \varepsilon, \ldots, x'_n) > u$$
   by continuity of $u$. Because $u(\tilde{x}) \geq u$, $\tilde{x}$ is feasible for $\min_x p \cdot x$ subject to $u(x) \geq u$. However,
   $$p\tilde{x} = px' - p_i\varepsilon < p \cdot x' = p \cdot x = e(p, u),$$
   which is a contradiction to the fact that $x'$ is the minimizer. Hence, we must have $e(p, u') > e(p, u)$.

   ii) Unbounded:
   Since $u(\cdot)$ is increasing and continuous, the image $\mathcal{U} = [u(0), \bar{u})$ where $\bar{u}$ can either be finite or infinite.[5] We would like to show that $e(p, u_n) \to \infty$ as $u_n \to \bar{u}$. We use the fact that the constraint binds at the optimum, i.e., $u(x^n) = u^n$.

   Consider a sequence $u^n \to \bar{u}$. Let $p \cdot x^n = e(p, u^n)$. For a contradiction, suppose $p \cdot x^n = e(p, u^n)$ is bounded. This implies that $x^n$ must be bounded, which implies that it has a convergent subsequence $\{x^n_k\}_{k=1}^{\infty}$ such that $x^n_k \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n_+$. Since $u$ is continuous, we must have $u(x^n_k) \equiv u_{n_k} \to u(\bar{x})$.

   Note that $u_{n_k}$ is a convergent subsequence of $u_n$, and must therefore converge to $\bar{u}$. Then, we have $u(\bar{x}) = \bar{u}$. Since $u$ is strictly increasing, we must have $u(\bar{x} + 1) > \bar{u}$, which is a contradiction to the fact that $\bar{u}$ is the supremum of the image.

4. Nondecreasing in $p$

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4. The last inequality comes from 1.
5. Example: $u(x_1, x_2) = e^{-x_1 x_2} \Rightarrow \mathcal{U} = [-1, 0)$.

Let $p' > p$ and $u$ be fixed. Let $x \in \mathbb{R}^n_+$ be the solution to the following problem:

$$\min_{x\in\mathbb{R}^n_+} p \cdot x \text{ s.t. } u(x) \geq u \qquad (1)$$

and let $x' \in \mathbb{R}^n_+$ be the solution to the following problem:

$$\min_{x\in\mathbb{R}^n_+} p' \cdot x \text{ s.t. } u(x) \geq u.$$

Then, we must have $e(p, u) = p \cdot x$ and $u(x) \geq u$, and $e(p', u) = p' \cdot x'$ and $u(x') \geq u$.

Since $u(x') \geq u$, $x'$ is feasible for the optimization in (1). Then, we must have $p \cdot x' \geq p \cdot x$, as $p \cdot x$ is the minimum of this program. Also, since $p' > p \gg 0$, we must have $p' \cdot x' > p \cdot x'$.

Together, we have $p' \cdot x' > p \cdot x \Rightarrow e(p', u) > e(p, u)$ as desired.

5. 1-homogeneity

$$x(tp, u) = \arg\min_{u(x)\geq u} tp \cdot x = \arg\min_{u(x)\geq u} p \cdot x = x(p, u),$$

i.e., scaling by a constant does not change the optimization problem.

$$\Rightarrow e(tp, u) = tp \cdot x(tp, u) = tp \cdot x(p, u) = te(p, u).$$

6. Concavity

Consider $p^0, p^1 \gg 0$ and let $\bar{p} = tp^0 + (1-t)p^1$. It suffices to show that

$$e(\bar{p}, u) \geq te(p^0, u) + (1-t)e(p^1, u).$$

Let $\bar{x} := x^h(\bar{p}, u), x^0 := x^h(p^0, u), x^1 := x^h(p^1, u)$. $p^0 \cdot x^0 = e(p^0, u), p^1 x^1 = e(p^1, u)$.
Note that $\bar{x}$ is feasible for the minimization problem under $p^0$ and $p^1$ and it solves $\min_x \bar{p} \cdot x$ subject to $u(x) \geq u$, i.e., we must have $u(\bar{x}) \geq u$. This implies

$$p^0 \cdot x^0 \leq p^0 \cdot \bar{x}, \;\; p^1 \cdot x^1 \leq p^1 \cdot \bar{x}$$

since $x^0$ and $x^1$ are minimizers in each problem. Then, we have

$$\begin{cases} tp^0 \cdot x^0 \leq tp^0 \cdot \bar{x} \\ (1-t)p^1 \cdot x^1 \leq (1-t)p^1 \cdot \bar{x} \end{cases} \Rightarrow te(p^0, u) + (1-t)e(p^1, u) \leq \bar{p} \cdot \bar{x} = e(\bar{p}, u),$$

as desired.[6]

7. Shephard's lemma

   i) Proof using envelope theorem.

$$e(p,u) = \mathcal{L}(x^*, \lambda^*) = p \cdot x^* - \lambda^* [\underbrace{u(x^*) - u}_{=0 \text{ at the solution}}]$$

$$\Rightarrow \frac{\partial e(p,u)}{\partial p_i} = x_i^h(p,u).$$

   ii) Alternative proof.
   Consider the following function:

$$f(p) := e(p, u_0) - p \cdot x_0,$$

   where $x_0 = x^h(p_0, u_0)$. Note that, by definition, $f(p_0) = 0$ and $f(p) \le 0, \forall p \ge 0$ since $e(p, u_0)$ is the minimum. This implies that $f$ is maximized at $p_0$. Then,

$$\frac{\partial f(p_0)}{\partial p_i} = 0 \iff \frac{\partial e(p_0, u_0)}{\partial p_i} - x_{0i} = 0$$

$$\Rightarrow x_i = \frac{\partial e(p,u)}{\partial p_i},$$

   as desired.

□

**Example 3.** Let $u(x_1, x_2) = x_1 x_2$. The expenditure generated by this utility would be obtained by solving the following problem:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \text{ subject to } x_1 x_2 \ge u.$$

Using the Lagrangian method,

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda(u - x_1 x_2).$$

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda x_2 = 0, \quad \frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda x_1 = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = u - x_1 x_2 = 0$$

6. Note that $e(p,u)$ is actually a support function – MWG for further reading.

$$\Rightarrow x_1^h = \left(\frac{p_2}{p_1}u\right)^{\frac{1}{2}}, x_2^h = \left(\frac{p_1}{p_2}u\right)^{\frac{1}{2}}$$

$$e(p_1, p_2, u) = p_1 \cdot \left(\frac{p_2}{p_1}u\right)^{\frac{1}{2}} + p_2 \cdot \left(\frac{p_1}{p_2}u\right)^{\frac{1}{2}} = 2(p_1 p_2 u)^{\frac{1}{2}}$$

---

#### 1.1.5 Duality

How do indirect utility and expenditure functions relate?

Let $u = V(p, I)$, then by definition, $e(p, u) \leq I$ as $e(\cdot)$ is the minimum for a given utility level $u$, which is feasible with income $I$.

Let $I = e(p, u)$, then by definition, $V(p, I) \geq u$ as $v(\cdot)$ is the maximum for a given income $I$, which can afford $u$.

$$\Rightarrow \begin{cases} e(p, V(p, I)) \leq I \\ V(p, e(p, u)) \geq u \end{cases}$$

**Theorem 7.** *If $u(\cdot)$ is continuous, strictly increasing, then $e(p, V(p, I)) = I$, $V(p, e(p, u)) = u, \forall p \gg 0, I \geq 0, u \in \mathcal{U}$.*

*Proof.*

1. Suppose $e(p, V(p, I)) \neq I$. Then, from the discussion above, $e(p, V(p, I)) < I$. Let $V(p, I) = u$, and thus $e(p, u) < I$. Since $e(p, u)$ is continuous in $(p, u)$, there exists $\varepsilon > 0$ such that $e(p, u + \varepsilon) < I$. Let $I_\varepsilon := e(p, u + \varepsilon)$. Then,

$$V(p, I_\varepsilon) \geq u + \varepsilon \Rightarrow u + \varepsilon \leq V(p, I_\varepsilon) < V(p, I) = u,$$

as $V$ is strictly increasing in $I$. In turn, we have $\varepsilon < 0$, which is a contradiction.

2. Suppose $V(p, e(p, u)) \neq u$. Then, from the discussion above, $V(p, e(p, u)) > u$. Let $e(p, u) = I$, and thus $V(p, I) > u$.

   (a) If $u = u(0)$, then $e(p, u) = 0$.

   In turn,

   $$V(p, 0) = 0 = u(0) \Rightarrow V(p, I) = u$$

   (b) If $u > u(0)$, then $e(p, u) > 0$. Since $V(p, I)$ is continuous in $(p, I)$, there exists $\varepsilon > 0$

such that $V(p, I - \varepsilon) > u$. Let $u_\varepsilon = V(p, I - \varepsilon)$. Note that $I - \varepsilon$ is sufficient income to achieve $u$. $e(p, u_\varepsilon) \leq I - \varepsilon$. Since $e(p, u)$ is strictly increasing in $u$, we have

$$I = e(p, u) < e(p, u_\varepsilon) = I - \varepsilon,$$

which is a contradiction.

□

We just proved a duality result for the value functions of the utility maximization and the expenditure minimization problems.

**Theorem 8** (Duality result for the solutions)**.** *Assume that $u(\cdot)$ is continuous, strictly increasing, and strictly quasiconcave. Then, we have the following relation between the Hicksian and the Marshallian demand.*

1. $x_i^*(p, I) = x_i^h(p, V(p, I))$
2. $x_i^h(p, u) = x_i^*(p, e(p, u))$

*Proof.*

1. By continuity and strict quasiconcavity of $u(\cdot)$, $x^*(p, I)$ is the unique solution to

$$\max_{x \in \mathbb{R}_+^n} u(x) \text{ s.t. } p \cdot x \leq I \qquad (\star)$$

To prove the theorem, it suffices to show that $x_i^h(p, V(p, I))$ solves $(\star)$. By the previous theorem.,

$$p \cdot x^h(p, V(p, I)) = e(p, V(p, I)) = I$$

That is, $x^h(p, V(p, I))$ is feasible for program $(\star)$. Moreover, $u(x^h(p, V(p, I)) \geq V(p, I)$ because $x^h(p, V(p, I))$ solves

$$\min_x p \cdot x \text{ s.t. } u(x) \geq V(p, I)$$

and hence must satisfy the constraint. Together, $x^h$ is feasible and yields utility of at least $V(p, I)$. Hence, $x^h$ must solve $(\star)$. Since the solution is unique given quasiconcavity, $x^*(p, I) = x^h(p, V(p, I))$.

2. (Analogous)

By continuity and strict quasiconcavity of $u(\cdot)$, $x^h(p,u)$ is the unique solution to

$$\min_{x \in \mathbb{R}^n_+} p \cdot x \text{ s.t. } u(x) \geq u \qquad (\star\star)$$

To prove the theorem, it suffices to show that $x^*(p, e(p,u))$ solves $(\star\star)$. By the previous theorem,

$$u(x^*(p, e(p,u))) = V(p, e(p,u)) = u.$$

Hence, $x^*(p, e(p,u))$ is feasible for the program $(\star\star)$. Moreover, we must have $p \cdot x^*(p, e(p,u)) \leq e(p,u)$ as $x^*(p, e(p,u))$ is the solution to the following problem:

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ s.t. } p \cdot x \leq e(p,u),$$

i.e., $x^*(p, e(p,u))$ must satisfy the constraint. As $e(p,u)$ is the minimum value, we must have $p \cdot x^* = e(p,u)$, which implies that $x^*(p, e(p,u))$ must solve $(\star\star)$. Since the solution is unique given quasiconcavity, $x^h(p,u) = x^*(p, e(p,u))$.

□

### 1.1.6 Testable Implications of Classical Consumer Theory

The implications of our theory are reflected on the properties of demands.

**Theorem 9** (Homogeneity of Degree Zero)**.** *Assume $u(\cdot)$ is continuous, strictly increasing and strictly quasiconcave. Then, the consumer's demand function is homogeneous of degree zero in prices and income, i.e.,*

$$x(tp, tI) = x(p, I), \forall t > 0$$

*and it satisfies budget balance (i.e., $p \cdot x(p, I) = I$, the Walras' law).*

*Proof.* Budget balancedness is immediate since $u(\cdot)$ is strictly increasing and thus the constraint binds. We have already proven that the indirect utility function is homogeneous of degree zero, i.e.,

$$V(tp, tI) = V(p, I) \Rightarrow u(x(tp, tI)) = u(x(p, I)).$$

This yields $x(tp, tI) = x(p, I)$ because $u(\cdot)$ is strictly quasiconcave and strictly increasing.

Alternatively, $B(p, I) = \{x : p \cdot x \leq I\} = \{x : tp \cdot x \leq tI\}$ so whenever $x(p, I)$ is feasible, so is $x(tp, tI)$ and vice versa. □

Changes in prices and income do not affect our consumer so long as they are proportional.

Next, we look at a change of a good's price holding everything else fixed (*ceteris paribus*).

Our theory does not necessarily predict that demand is downward sloping. A good whose demand is upward-sloping is called a Giffen good. The reason why a Marshallian demand can be upward sloping is related to how changes in income affect choices. We can classify goods based on how changes in income affects their demand.

A simple taxonomy is as follows:

- a good is normal at $(p^0, I^0)$[7] if $\frac{\partial x_i(p^0,I^0)}{\partial I} \geq 0$.
- a good is inferior at $(p^0, I^0)$ if $\frac{\partial x_i(p^0,I^0)}{\partial I} < 0$.
- a Giffen good at $(p^0, I^0)$ satisfies $\frac{\partial x_i(p^0,I^0)}{\partial p_i} \geq 0$.

Recall that $x^h(p, u)$ is always downward sloping, because it captures only changes in relative prices and the resulting moves along the same indifference curve. (The income effect is not there as utility level is fixed.)

Conclusion: a Giffen good must necessarily be inferior.

$$\underbrace{\frac{\partial x_i(p^0, I^0)}{\partial p_i}}_{\text{Effect of price change}} = \underbrace{\text{substitution effect}}_{\text{(SE)}} + \underbrace{\text{income effect}}_{\text{(IE)}}$$

**Theorem 10** (Slutsky's Decomposition)**.**

$$\frac{\partial x_i(p^0, I^0)}{\partial p_j} = \underbrace{\frac{\partial x_i^h(p^0, u^0)}{\partial p_j}}_{\text{(SE)}} - \underbrace{x_j(p^0, I^0)\frac{\partial x_i(p^0, I^0)}{\partial I}}_{\text{(IE)}}$$

*for any* $i, j$ *where* $u^0 = V(p^0, I^0)$.

*Proof.* We have shown that

$$x_i^h(p, u) = x_i(p, e(p, u)), \forall p \gg 0, e(p, u).$$

Taking derivatives of each side yields

$$\frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial x_i(p, e(p, u))}{\partial p_j} + \frac{\partial x_i(p, e(p, u))}{\partial I} \cdot \frac{\partial e(p, u)}{\partial p_j}$$
$$\Rightarrow \frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial x_i(p, e(p, u))}{\partial p_j} + x_j^h(p, u)\frac{\partial x_i(p, e(p, u))}{\partial I}$$

7. Always relative to $p$, $I$.

where the last step made use of Shephard's lemma. Then, at $(p^0, I^0)$, we have

$$\begin{aligned}\frac{\partial x_i(p^0, I^0)}{\partial p_j} &= \frac{\partial x_i^h(p,u)}{\partial p_j} - x_j^h(p,u)\frac{\partial x_i(p, e(p,u))}{\partial I} \\ &= \frac{\partial x_i^h(p,u)}{\partial p_j} - x_j(p,I)\frac{\partial x_i(p, e(p,u))}{\partial I}\end{aligned}$$

by duality, as desired. □

**Theorem 11.** *Consider the substitution matrix, defined by*

$$\sigma(p,u) \equiv \left(\frac{\partial x_i^h(p,u)}{\partial p_j}\right)_{1\le i,j\le n}$$

*If $e(p,u)$ is twice differentiable in $p$, then this matrix is symmetric, negative semi-definite.*

*Remark.* Negative semi-definiteness implies that $x_i^h$ is downward sloping. Take $y = e_i$, the $i$th standard basis vector, then we must have $y^\top \sigma(p,u) y \le 0, \forall i$.

*Proof.*

1. Symmetry
   Recall that $x_i^h(p,u) = \frac{\partial e(p,u)}{\partial p_i}$ by Shephard's lemma. Then,

$$\begin{aligned}\sigma_{ij} = \frac{\partial x_i^h(p,u)}{\partial p_j} &= \frac{\partial^2 e(p,u)}{\partial p_j \partial p_i}, \forall i,j \\ &= \frac{\partial^2 e(p,u)}{\partial p_i \partial p_j} \text{(by Young's theorem.)} \\ &= \frac{\partial x_j^h(p,u)}{\partial p_i} = \sigma_{ji}, \forall i,j\end{aligned}$$

   Hence, $\sigma(p,u)$ is symmetric.

2. Negative semi-definiteness
   $\sigma(p,u)$ is n.s.d. if and only if $e(p,u)$ is concave. We have shown the latter, so we are done.

□

Going back to Slutsky decomposition,

$$\underbrace{\frac{\partial x_i^h(p,u)}{\partial p_j}}_{\sigma(p,u)} = \underbrace{\frac{\partial x_i(p,I)}{\partial p_j} + x_j(p,I)\frac{\partial x_i(p,I)}{\partial I}}_{S(p,I)}$$

**Theorem 12.** *Let*

$$S(p,I) = \left( \frac{\partial x_i(p,I)}{\partial p_j} + x_j(p,I)\frac{\partial x_i(p,I)}{\partial I} \right)_{1\le i,j\le n}.$$

*The Slutsky matrix, $S(p,I)$, is symmetric, negative semi-definite, so long as $e(p,u)$ is twice differentiable in $p$.*

Note that this is basically the same as the previous theorem.

Hence, our observable implications are:

1. Demand is homogeneous of degree zero.
2. Walras law
3. $S(p,I)$: symmetric
4. $S(p,I)$: n.s.d.

Do these four conditions exhaust the observable implications of consumer theory? – Yes.

For any function $x(p,I)$ satisfying the above four conditions, there exists a utility function generating it.

How to derive a utility function from an expenditure function.
Suppose we have a function $e(p,u)$ on $\underbrace{\mathbb{R}^n_{++}}_{p} \times \underbrace{\mathbb{R}_+}_{u}$ satisfying properties 1-7 of Theorem 6.

Note that $\forall x, \forall p \gg 0$, $p\cdot x \ge e(p,u(x))$. This inequality holds as $x$ suffices to achieve $u(x)$ and $e(p,u)$ is the minimum value of the program $\min_{x\in\mathbb{R}^n_+} p\cdot x$ s.t. $\tilde{u}(x) \ge u(x)$.

Suppose $p^0\cdot x = e(p^0,u(x))$. Then, $p^0\cdot x < e(p,u)$ for $u > u(x)$. Therefore, if $u > u(x)$, then the condition $p\cdot x \ge e(p,u)$ fails at $p = p^0$ because $e(p,u)$ is strictly increasing in $u$. We can construct the utility as

$$u(x) := \max\{u : p\cdot x \ge e(p,u), \forall p \gg 0\}.$$

To see that the maximum exists, consider the set

$$A(p,u) = \{x \in \mathbb{R}^n_+ : p\cdot x \ge e(p,u)\}.$$

We can trace the utility curve and obtain the at-least-as-good-as set by taking an intersection over all $p \gg 0$, i.e.,

$$A(u) = \bigcap_{p\gg 0} A(p,u).$$

**Theorem 13.** *Let $e : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying the 7 properties of Theorem 6.*

*The function* $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ *defined by*

$$u(x) := \max\{u : p \cdot x \geq e(p, u), \forall p \gg 0\}$$

*is well-defined, increasing, unbounded, and quasiconcave.*

*Proof.*

i) Well-defined
Consider $x \in \mathbb{R}^n_+$. Define

$$\mathcal{U}(x) = \{u \geq 0 : p \cdot x \geq e(p, u), \forall p \gg 0\}.$$

Note first that $\mathcal{U}(x)$ is nonempty because $0 \in \mathcal{U}(x)$. It is also bounded below by 0 and bounded above $x \in \mathbb{R}^n_+$ (so $p \cdot x$ is bounded) and $e(p, u)$ is strictly increasing and unbounded above in $u$.

Hence, when $e(p, u)$ is below a finite number, the $u$ that gives that bound is finite. So $\mathcal{U}(x)$ is bounded.

In fact, $\mathcal{U}(x)$ is also closed. Consider a sequence $u_n \in \mathcal{U}(x)$ such that $u_n \to u$. Note that

$$p \cdot x \geq e(p, u_n), \forall p \gg 0, \forall n.$$

Moreover, $e(p, u_n) \to e(p, u)$ by continuity of $e(p, \cdot)$, which implies $p \cdot x \geq e(p, u), \forall p \gg 0$. In turn, we have $u \in \mathcal{U}(x)$. Hence, $\mathcal{U}(x)$ is closed.

From the above, we have $\mathcal{U}(x)$ is compact. Therefore, $u(x)$ is well-defined and continuous.

ii) Increasing
Consider $x^1 \geq x^2$. By definition, $p \cdot x^1 \geq p \cdot x^2 \geq e(p, u(x^2))$. Recall that $u(x^1)$ is defined as the maximum $u$ such that $u(x^1) = \max\{u \geq 0 : p \cdot x^1 \geq e(p, u), \forall p \gg 0\}$. Hence, $u(x^1) \geq u(x^2)$. (Essentially $u(x^1)$ is a maximum over a bigger set.)

iii) Quasiconcavity
Let $x^1, x^2 \in \mathbb{R}^n_+$. Without loss of generality, assume $\min\{u(x^1), u(x^2)\} = u(x^1)$. Note that

$$p \cdot x^1 \geq e(p, u(x^1))$$
$$p \cdot x^2 \geq e(p, u(x^2)$$

Let $\bar{x} = tx^1 + (1-t)x^2$ for some $t \in [0,1]$. Then,

$$\begin{aligned} p \cdot \bar{x} &\geq te(p, u(x^1)) + (1-t)e(p, u(x^2)) \\ &\geq te(p, u(x^1)) + (1-t)e(p, u(x^1)) \\ &= e(p, u(x^1)) \end{aligned}$$

By definition,

$$u(\bar{x}) = \max\{u \geq 0 : p \cdot \bar{x} \geq e(p,u), \forall p \gg 0\}.$$

Then, $u(\bar{x}) \geq u(x^1)$, since $p \cdot \bar{x} \geq e(p, u(x^1))$.

iv) Unbounded

Suppose $u(\cdot)$ is bounded. Then, there exists $\bar{u}$ such that $u(x) \leq \bar{u}, \forall x \in \mathbb{R}^n_+$. Then, we have $e(p, u(x)) \leq e(p, \bar{u}), \forall x \in \mathbb{R}^n_+$ as $e(p, \cdot)$ is strictly increasing in $u$. By homogeneity of degree 1 in $p$, we have

$$e(p,u) = \frac{\partial e(p,u)}{\partial p} \cdot p, \quad \forall p \gg 0.$$ [a]

Substituting yields

$$\frac{\partial e(p, u(x))}{\partial p} \cdot p \leq \frac{\partial e(p, \bar{u})}{\partial p} \cdot p, \quad \forall p \gg 0, \forall x \in \mathbb{R}^n_+.$$

Then, for $\varepsilon > 0$, we can define $x^0 := \frac{\partial e(p,\bar{u})}{\partial p} + \varepsilon \mathbb{1}_n$. For this particular $x^0$, we have

$$\frac{\partial e(p, u(x^0))}{\partial p} \cdot p \leq \frac{\partial e(p, \bar{u})}{\partial p} \cdot p < p \cdot x^0 \Rightarrow e(p, u(x^0)) < p \cdot x^0.$$

Then, for some $\delta > 0$, we must have $e(p, u(x^0) + \delta) \leq p \cdot x^0$, which is a contradiction to the fact that $u(x^0)$ is the maximum. Hence, $u(x)$ must be unbounded above.

*a*. See proof of Theorem 14.

□

**Theorem 14.** *Suppose* $e(\cdot,\cdot)$ *satisfies the 7 properties of Theorem 6. Then,* $\forall p \gg 0, u \geq 0$,

$$e(p,u) = \min_{x \in \mathbb{R}^n_+} p \cdot x \text{ s.t. } u(x) \geq u.$$

*Proof.* Fix $p^0 \gg 0, u^0 \geq 0$ and $x \in \mathbb{R}^n_+$ such that $u(x) \geq u^0$. It is possible to do so because $u(\cdot)$

defined in the previous theorem is increasing and unbounded.

By definition, $u(x)$, $p \cdot x \geq e(p, u(x)), \forall p \gg 0$. Also, because $e$ is increasing in $u$ and $u(x) \geq u^0$, $p \cdot x \geq e(p, u^0), \forall p \gg 0$. We have shown that $p^0 \cdot x \geq e(p^0, u^0), \forall x \in \mathbb{R}^n_+$ with $u(x) \geq u^0$. Then, $e(p^0, u^0) \leq \min_{x \in \mathbb{R}^n_+} p^0 \cdot x$ subject to $u(x) \geq u^0$.

It remains to show that there exists $x^0 \in \mathbb{R}^n_+$ with $u(x^0) \geq u^0$ and $p^0 \cdot x^0 \leq e(p^0, u^0)$.

Recall that $e(\cdot, \cdot)$ is increasing in $p$. Thus,

$$\frac{\partial e}{\partial p} = \left( \frac{\partial e}{\partial p_1}, \ldots, \frac{\partial e}{\partial p_n} \right) \geq 0.$$

Also, $e(p, u)$ is homogeneous of degree 1 in $p$ implies

$$\forall p \gg 0, e(p, u^0) = p \cdot \frac{\partial e(p, u^0)}{\partial p}.$$

Since $e(p, u)$ is homogeneous of degree 1 in $p$, we have

$$e(tp, u) - te(p, u), \forall p \gg 0.$$

Differentiating the above with respect to $t$ and evaluating at $t = 1$ gives

$$\sum_{i=1}^{n} \frac{\partial e(p, u)}{\partial p_i} p_i - e(p, u) = 0 \Rightarrow e(p, u) = p \cdot \frac{\partial e(p, u)}{\partial p}$$

as desired.

Moreover, $e(\cdot, \cdot)$ is concave in $p$, $u$. Concavity implies

$$e(p, u) \leq e(p^0, u) + \frac{\partial e(p^0, u)}{\partial p} \cdot (p - p^0).$$

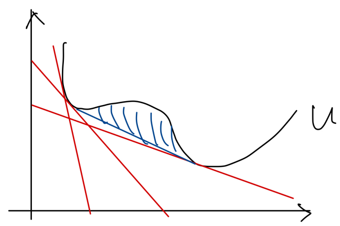
Evaluating the above at $u^0$ and combining with the previous equation yields

$$e(p^0, u^0) \geq \frac{\partial e(p, u^0)}{\partial p} \cdot p^0.$$

Let $x^0 = \frac{\partial e(p, u^0)}{\partial p} \geq 0$. Then, we have $e(p^0, u^0) \geq x^0 \cdot p^0$ as desired. Further note that if $u(x^0) < u^0$, this would imply $e(p, u(x^0)) < e(p, u^0) = p \cdot x^0$ as $e(p, \cdot)$ is strictly increasing. This is a contradiction, which implies that $x^0$ must satisfy the constraint of the minimization problem. □

Take any $u(\cdot)$ that is neither quasiconcave nor increasing. This function will generate the same expenditure function as another utility function that is both increasing and quasiconcave. An illustration:

Figure 1: Preference that Violates Convexity and Monotonicity




Note that in Figure 1, the indifference curve indicates that the underlying preference violates convexity and monotonicity. However, when minimizing expenditure, the agent will behave as if convexity is satisfied (i.e., as if the region shaded in blue was filled). The only exception is when the price vector coincides with the blue line, which is basically an ignorable probability (i.e., a measure-zero event). From this, we learn that continuity and transitivity are the key axioms that drive predictions.

**Theorem 15.** *If $x(p, I)$ satisfies budget balance and symmetry of the Slutsky matrix, then it is homogeneous of degree 0 in $(p, I)$.*

*Proof.* Budget balancedness implies $p \cdot x(p, I) = I, \forall p, I$, which can be explicitly written as

$$\sum_{j=1}^{n} p_j x_j(p, I) = I.$$

Differentiating with respect to $p_i$ yields:

$$\sum_{j=1}^{n} p_j \frac{\partial x_j(p, I)}{\partial p_i} + x_i(p, I) = 0 \quad (1)$$

and differentiating with respect to $I$ yields:

$$\sum_{j=1}^{n} p_j \frac{\partial x_j(p, I)}{\partial I} = 1. \quad (2)$$

Fix $p, I$ and for each $i$, let

$$f_i(t) = x_i(tp, tI), \forall t > 0.$$

It suffices to show that $f_i'(t) = 0, \forall t > 0$.

$$\begin{aligned} f_i'(t) &= \sum_{j=1}^{n} \frac{\partial x_i(tp, tI)}{\partial p_j} p_j + \frac{\partial x_i(tp, tI)}{\partial I} I \\ &= \sum_{j=1}^{n} p_j \left[ \frac{\partial x_i(tp, tI)}{\partial p_j} + \frac{\partial x_i(tp, tI)}{\partial I} x_j(tp, tI) \right] \end{aligned}$$

By symmetry, we have

$$\begin{aligned} f_i'(t) &= \sum_{j=1}^{n} \frac{1}{t} p_j \left[ t \frac{\partial x_j(tp, tI)}{\partial p_i} + t x_i(tp, tI) \frac{\partial x_j(tp, tI)}{\partial I} \right] \\ &= \frac{1}{t} [\underbrace{-t x_i(tp, tI)}_{\text{by (1)}} + \underbrace{t x_i(tp, tI)}_{\text{by (2)}}] = 0 \end{aligned}$$

as desired. □

**Theorem 16** (Integrability Theorem)**.** *A continuously differentiable function* $x : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^n_+$ *is a demand function generated by some increasing and quasiconcave utility function if and only if it satisfies budget balancedness, symmetry and negative semi-definiteness of* $S(p, I)$*.*

*Proof.*

(⇐) Done. (The derivation of Marshallian demand.)

(⇒) Take some expenditure function $e(p, u)$ generated by some utility function that is increasing and quasiconcave. Let $x^*(p, I)$ denote the Marshallian demand for some $u(\cdot)$.

Suppose that $x$ and $e$ are related as follows:

$$\frac{\partial e(p, u)}{\partial p_i} = x_i(p, e(p, u)), \forall p, u, i \quad (1)$$

If (1) is true, then $x_i(p, I) = x_i^*(p, I)$ (by Shephard's lemma). Now, we would like to know when the solutions to the system of PDEs in (1) exist.

Suppose a solution exists. Then, differentiating (1) with respect to $p_j$ yields

$$\begin{aligned} \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} &= \frac{\partial x_i(p, e(p, u))}{\partial p_j} + \frac{\partial x_i(p, e(p, u))}{\partial I} \frac{\partial e(p, u)}{\partial p_j} \\ \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} &= \frac{\partial x_i(p, I)}{\partial p_j} + x_j(p, I) \frac{\partial x_i(p, I)}{\partial I}. \end{aligned}$$

Young's theorem implies that, if a solution exists, $\left( \frac{\partial^2 x_i(p,I)}{\partial p_j \partial p_i} \right)$ must be symmetric. Hence, a necessary condition for system (1) to have a solution is the symmetry of $S(p, I)$. Frobenius theorem says this is also a sufficient condition.

It remains to show that the solution is indeed an expenditure.

□

**Example 4.** Consider 3 goods economy with the following demand:

$$x_i(p_1, p_2, p_3, I) = \frac{a_i I}{p_i}, \quad \sum_i a_i = 1, \quad i = 1, 2, 3.$$

Then, the system becomes

$$\frac{\partial e(p, u)}{\partial p_i} = a_i \frac{e(p, u)}{p_i}, \quad i = 1, 2, 3.$$

Solving this system gives

$$\ln e(p_1, p_2, p_3, u) = a_1 \ln p_1 + a_2 \ln p_2 + a_3 \ln p_3 + c(u) \Rightarrow e(p_1, p_2, p_3, u) = c(u) p_1^{a_1} p_2^{a_2} p_3^{a_3}.$$

(This is the Cobb-Douglas function.)

## 1.2 Revealed Preference Theory

We have relied on the existence of "preference" so far, but now we look at choices instead. We are still studying consumer behavior but just changing the primitive from preferences to choices (i.e., what's observed).

The primitive of choice is the choice function:

$$x : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+, \;\; p \in \mathbb{R}^n_{++}, I \in \mathbb{R}_+.$$

We will axiomatize choices. The next axiom is a mild condition on the choice function to capture the notion that a bundle $x_0$ has been revealed preferred to another bundle $x_1$.

### 1.2.1 Weak Axiom of Revealed Preference

**Definition 5** (Weak axiom of revealed preference; WARP)**.** Consumer choice behavior satisfies WARP if, for every pair of distinct bundles $x^0 \neq x^1$, $x^0$ is chosen at $p^0$ and $x^1$ is chosen at price $p^1$, then $p^0 \cdot x^1 \leq p^0 \cdot x^0 \Rightarrow p^1 \cdot x^0 > p^1 \cdot x^1$.

In short,

$$\forall (p, I), (p', I'), \;\; p \cdot x(p', I') < I, \;\; x(p, I) \neq x(p', I') \Rightarrow p' \cdot x(p, I) > I'.$$

If $p^0 \cdot x^1 \leq p^0 \cdot x^0$, then $x^1$ was affordable at prices $p^0$ when $x^0$ was chosen. This means that if the consumer is choosing $x^1$, then they could not have afforded $x^0$ at $p^1$.

Marshallian demand, viewed as a choice function, will satisfy WARP. Note that $u(x^0) > u(x^1) \iff x^0 \succ x^1$. This means that $x^1$ will never be chosen when $x^0$ is available. Thus, $x^0 \in B(p, I)$ will imply that the consumer will choose $x^0$ over $x^1$. On the other hand, if $x^1$ is chosen, it must be the case that $x^0 \notin B(p, I)$. Hence, viewed as a choice function, Marshallian demand satisfies WARP.

Now, we impose one more condition on $x(p, I)$. Suppose it also satisfies budget balance: $p \cdot x(p, I) = I$. What are the implications of WARP and budget balance?

**Theorem 17.** *WARP and budget balance imply homogeneity of degree zero.*

*Proof.* Let $x^0 = x(p^0, I^0)$, $x^1 = x(tp^0, tI^0)$. Then, by budget balancedness,

$$p^0 \cdot x^0 = I^0 = t\frac{1}{t}I^0,$$

$$p^1 \cdot x^1 = tI^0 \iff tp^0 \cdot x^1 = tI^0.$$

Together, we have

$$p^0 \cdot x^0 = \frac{1}{t}(tp^0 \cdot x^1) \iff tp^0 \cdot x^0 = tp^0 \cdot x^1 = tI^0 \iff \begin{cases} p^1 \cdot x^0 = I^1 \\ p^0 \cdot x^1 = I^0 \end{cases}.$$

So $x^0$ is chosen when $x^1$ is affordable at $(p^0, I^0)$. At the same time, $x^1$ was chosen when $x^0$ is affordable at $(tp^0, tI^0)$. Thus, by WARP, we must have $x^0 = x^1$. □

Is the choice function $x(p, I)$ that satisfies WARP and budget balance (and thus HD0) actually a demand function (i.e., generated from utility)? The answer is yes if the Slutsky matrix is symmetric, negative semi-definite. To investigate, we work with the Slutsky compensated choice function.

Let $x^0 = x(p^0, I^0)$ and consider the choices made by the consumer when prices vary but her income is compensated so she can still afford $x^0$: $x(p, p \cdot x^0)$.

Consider an arbitrary price level $p^1$ and let $x^1 = x(p^1, p^1 \cdot x^0)$. By budget balancedness, $p^1 \cdot x^1 = p^1 \cdot x^0$. $x^1$ was chosen when $x^0$ was available. If they are distinct bundles ($x^1 \neq x^0$), by WARP, $p^0 \cdot x^1 > p^0 \cdot x^0$. When $x^1 = x^0$, then we would just have $p^0 \cdot x^1 \geq p^0 \cdot x^0$. In either case, we must have

$$(p^1 - p^0) \cdot x^1 \geq (p^1 - p^0) \cdot x^0, \forall p^1 \gg 0.$$

We can write $p^1 \equiv p^0 + tz$ for some $t \in \mathbb{R}_+$, $z \in \mathbb{R}^n$. $p^1 \gg 0$ by choosing $t$ very small if $z$ has negative elements. Then, $tz \cdot x(p^1, p^1 \cdot x^0) \leq tz \cdot x^0 \iff z \cdot x(p^0 + tz, (p^0 + tz) \cdot x^0) \leq z \cdot x^0$. Note that the LHS is a function of $t$.

$$f(t) = z \cdot x(p^0 + tz, (p^0 + tz) \cdot x^0)$$

is maximized at $t = 0$. Therefore, $f'(0) \leq 0$.

$$\Rightarrow f(t) = \sum_{i=1}^{n} z_i x_i(p^0 + tz, (p^0 + tz) \cdot x^0)$$

$$\begin{aligned}\Rightarrow f'(t) &= \sum_{i=1}^{n} z_i \left[ \sum_{j=1}^{n} \frac{\partial x_i}{\partial p_j} z_j + \frac{\partial x_i}{\partial I} \sum_{j=1}^{n} z_j x_j^0 \right] \\ &= \sum_{i=1}^{n} z_i \left[ \sum_{j=1}^{n} z_j \left( \frac{\partial x_i}{\partial p_j} + x_j^0 \frac{\partial x_i}{\partial I} \right) \right] \\ &= z^\top S(p^0, I) z \leq 0\end{aligned}$$

Since $z \in \mathbb{R}^n$ was arbitrary, $S(p^0, I^0)$ is n.s.d.

Unfortunately, $S(p, I)$ is not symmetric. (Symmetry is only implied for two goods.) To get symmetry for $n \geq 2$, we need to strengthen the axiom (the Strong Axiom of Revealed Preference; SARP).

**Definition 6** (Strong Axiom of Revealed Preference; SARP)**.** A consumer's behavior satisfies SARP if whenever $x^k$ is revealed preferred to $x^{k+1}$, $x^k$ is chosen when $x^{k+1}$ is affordable, $k = 0, \ldots, n$, then $x^{k+1}$ is never chosen when $x^0$ is available. That is, if $x^0 R x^1 R \cdots R x^{k+1}$, then $x^{k+1} R x^0$ is not true (i.e., some notion of transitivity).

In short,

$$\forall (p_n, I_n), n = 1, \ldots, N, \;\; p_n \cdot x(p_{n+1}, I_{n+1}) < I_n, \;\; x(p_n, I_n) \neq x(p_{n+1}, I_{n+1}) \Rightarrow p_N \cdot x(p_1, I_1) > I_N$$

## 1.3 Choices Under Uncertainty

So far, we have worked with choices under certainty. We now consider gambles (uncertain situations).

### 1.3.1 Axiomatic Foundation of Expected Utility

Note that this is not very well-founded in behavior (and is subject to some controversy). Let $A$ be a finite set of outcomes

$$A = \{a_1, a_2, \ldots, a_n\}.$$

We consider lotteries whose ultimate outcomes are elements of $A$. A *simple lottery* is a lottery all of whose outcomes are elements of $A$. The set of simple lotteries is the following

$$\mathcal{G}_s \equiv \{(p_1 \circ a_1, \ldots, p_n \circ a_n) : p_i \geq 0, \sum_i p_i = 1\}, \mathcal{G}_s \equiv \Delta(A)$$

where $\Delta(A)$ is the set of all distributions over $A$. A *compound lottery* is a lottery that is not simple.

$$\mathcal{G}_1 = \Delta(\mathcal{G}_s) \rightsquigarrow (p_1 \circ g_1^s, \ldots, p_n \circ g_n^s),$$

i.e., a lottery of lotteries. We will assume that for each compound lottery some outcome in $A$ occurs with probability 1 after a finite number of randomizations.

Let $\mathcal{G}$ denote the set of all such lotteries. We will assume that the consumer has preferences over elements of $\mathcal{G}$, $\succsim$ on $\mathcal{G}$ satisfying the following axioms.

**Axiom G1** (Completeness). For all distinct gambles $g, g'$, either $g \succsim g'$ or $g' \succsim g$.

**Axiom G2** (Transitivity). $\forall g, g', g''$, if $g \succsim g'$ and $g' \succsim g''$, then $g \succsim g''$.

By G1-G2, $\succsim$ can rank the finite outcomes in $A$ and relabel, i.e., $a_1 \succsim a_2 \succsim \cdots \succsim a_n$ (which can be done as viewing each individual outcome as a degenerate lottery, or the Dirac delta distribution).

**Axiom G3** (Continuity). For every gamble $g \in \mathcal{G}$, there is a probability $\alpha$ such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n), \alpha \in [0, 1]$.

**Axiom G4** (Monotonicity). For all probabilities $\alpha, \beta \in [0, 1]$, $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ if and only if $\alpha \geq \beta$.

**Axiom G5** (Substitution). If $g = (p_1 \circ g^1, p_2 \circ g^2, \ldots, p_n \circ g^n)$, $h = (p_1 \circ h^1, p_2 \circ h^2, \ldots, p_n \circ h^n)$, $g, h \in \mathcal{G}$ and $g^i \sim h^i, \forall i = 1, \ldots, n$, then $g \sim h$.

**Axiom G6** (Reduction to Simple Gambles). If $g \in \mathcal{G}$ and $g_s = (p_1 \circ a_1, \ldots, p_n \circ a_n)$ induced by $g$ (i.e., the effective probability of actual outcomes), then $g \sim g_s$.

### 1.3.2 Von Neumann-Morgenstern Utility

**Definition 7** (Von Neumann-Morgenstern Utility). $u : \mathcal{G} \to \mathbb{R}$ satisfies the expected utility property if, for all $g \in \mathcal{G}$,

$$u(g) = \sum_{i=1}^{n} p_i u(a_i) = u(g_s)$$

where $g_s = (p_1 \circ a_1, \ldots, p_n \circ a_n)$ is the simple lottery induced by $g$. We also call such functions von Neumann-Morgenstern utility functions.

**Theorem 18.** *If $\succsim$ satisfies Axioms G1-G6, then there is a von Neumann-Morgenstern utility function representing $\succsim$.*

*Proof.* For ech gamble $g \in \mathcal{G}$, let $u(g)$ be defined as follows:

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n).$$

Such a number exists by continuity (G3) and is unique by monotonicity (G4). Note that $u(g) \in [0, 1]$. We first claim that $u(g)$ represents $\succsim$.

Suppose $g \succsim g'$. Then, $(u(g) \circ a_1, (1-u(g)) \circ a_n) \succsim (u(g') \circ a_1, (1-u(g')) \circ a_n)$ by transitivity (G2) and the definition of $u(g)$. By monotonicity (G4), $u(g) \geq u(g')$. Therefore, $g \succsim g' \iff u(g) \geq u(g')$. Hence, $u(\cdot)$ represents $\succsim$.

It remains to show that $u(\cdot)$ satisfies the expected utility property. Let $g \in \mathcal{G}$ be an arbitrary gamble and $g_s = (p_1 \circ a_1, \ldots, p_n \circ a_n)$ be the simple gamble that it induces. It suffices to show

$$u(g_s) = \sum_{i=1}^{n} p_i u(a_i),$$

since $g \sim g_s$ by substitution (G5). Define gambles $q_i$ as

$$a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n) =: q_i, \forall i = 1, \ldots, n$$

as each $a_i$ is a degenerate gamble. Note that we then have $g_s \sim \underbrace{(p_1 \circ q_1, \ldots, p_n \circ q_n)}_{=:g'}$. Then, $g'$ is a compound gamble with only $a_1$ and $a_n$ as its realizations. Let $g'_s$ be the simple gamble induced by $g'$. Then,

$$g' \sim g'_s = \left( \sum_{i=1}^{n} p_i u(a_i) \circ a_1, \sum_{i=1} p_i (1 - u(a_i)) \circ a_n \right) \Rightarrow u(g'_s) = \sum_{i=1}^{n} p_i u(a_i).$$

By transitivity,

$$u(g_s) = \sum_{i=1}^{n} p_i u(a_i)$$

as desired. □

**Theorem 19** (Uniqueness of vNM up to Affine Transformation)**.** *Suppose $u(\cdot)$ is a vNM representation of $\succsim$ over lotteries of $\mathcal{G}$, then $v(\cdot)$ is a vNM representation of $\succsim$ if and only if $v(g) = \alpha + \beta u(g), \alpha \in \mathbb{R}, \beta > 0$.*

*Proof.*

(⇒) (Adapted from Jehle and Reny (2010))

Let $u(\cdot), v(\cdot)$ be vNM representations of $\succsim$ over $\mathcal{G}$. Consider a simple gamble $g \in \mathcal{G}$, $g := (p_1 \circ a_1, \ldots, p_n \circ a_n)$, where $a_1 \succsim \cdots \succsim a_n$ and $a_1 \succ a_n$. By continuity, there exists $\alpha_i \in [0, 1]$ such that

$$u(a_i) = \alpha_i u(a_1) + (1 - \alpha_i) u(a_n), \ \ \forall i = 1, \ldots, n.$$

Since $u(\cdot)$ is a vNM utility, we have

$$u(a_i) = u(\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), \ \ \forall i.$$

$$\iff a_i \sim (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), \ \ \forall i$$

Since $v(\cdot)$ also represents $\succsim$, we must have

$$v(a_i) = v(\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), \ \ \forall i.$$

As $v(\cdot)$ is also a vNM utility, this implies

$$v(a_i) = \alpha_i v(a_1) + (1 - \alpha_i) v(a_n), \ \ \forall i.$$

Hence, we have

$$\begin{cases} \alpha_i u(a_i) + (1 - \alpha_i) u(a_i) = \alpha_i u(a_1) + (1 - \alpha_i) u(a_n) \\ \alpha_i v(a_i) + (1 - \alpha_i) v(a_i) = \alpha_i v(a_1) + (1 - \alpha_i) v(a_n) \end{cases}, \ \ \forall i$$

For $i$ with $a_i \succ a_n$, we have

$$\begin{cases} \frac{1-\alpha_i}{\alpha_i} = \frac{u(a_1) - u(a_i)}{u(a_i) - u(a_n)} \\ \frac{1-\alpha_i}{\alpha_i} = \frac{v(a_1) - v(a_i)}{v(a_i) - v(a_n)} \end{cases}$$

Rearranging gives

$$(u(a_1) - u(a_i))(v(a_i) - v(a_n)) = (v(a_1) - v(a_i))(u(a_i) - u(a_n))$$

Note that the above also holds for $a_i \sim a_n$, as we have $u(a_i) = u(a_n), v(a_i) = v(a_n)$ and both sides will equate to zero. Hence, the above holds for all $i = 1, \ldots, n$. Further rearranging gives

$$v(a_i) = \frac{u(a_1)v(a_n) - v(a_1)u(a_n)}{u(a_1) - u(a_n)} + \frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)} u(a_i)$$

Let $\alpha := \frac{u(a_1)v(a_n)-v(a_1)u(a_n)}{u(a_1)-u(a_n)}$ and $\beta := \frac{v(a_1)-v(a_n)}{u(a_1)-u(a_n)}$. Note that $\beta > 0$, since $a_1 \succ a_n$. Then,

$$\begin{aligned} v(g) &= \sum_{i=1}^{n} p_i v(a_i) \\ &= \sum_{i=1}^{n} p_i(\alpha + \beta u(a_i)) \\ &= \alpha \sum_{i=1}^{n} p_i + \beta \sum_{i=1}^{n} u(a_i) \\ &= \alpha + \beta u(g) \end{aligned}$$

which is the desired.

($\Leftarrow$) Let $u(\cdot)$ be a vNM representation of $\succsim$ over $\mathcal{G}$ and define $v(g) := \alpha + \beta u(g)$ for some $\alpha \in \mathbb{R}, \beta > 0$. Let $g, g' \in \mathcal{G}$. Then, we have the following

$$g \succsim g' \iff u(g) \geq u(g') \iff \alpha + \beta u(g) \geq \alpha + \beta u(g') \iff v(g) \geq v(g').$$

Hence, $v(\cdot)$ is a utility representation of $\succsim$.

Now, consider a simple gamble $g_s := (p_1 \circ a_1, \ldots, p_n \circ a_n)$. Since $u(\cdot)$ is a vNM representation, we have

$$u(g_s) = \sum_i p_i u(a_i) \iff \alpha + \beta u(g_s) = \alpha + \beta \sum_i p_i u(a_i)$$

$$\iff v(g_s) = \alpha \sum_i p_i + \beta \sum_i p_i u(a_i) \iff v(g_s) = \sum_i p_i(\alpha + \beta u(a_i))$$

$$\iff v(g_s) = \sum_i p_i v(a_i).$$

Hence, $v(\cdot)$ is a vNM representation.

□

Lotteries over money (continuous)

$$L = \begin{cases} \text{win } \$100 & \text{with prob. } \frac{1}{2} \\ \text{lose } \$100 & \text{with prob. } \frac{1}{2} \end{cases}$$

Expected value of $L$: 0. Expected utility of $L$: $\frac{1}{2}u(100) + \frac{1}{2}u(-100)$, which could be different from 0.

Expected utility representation captures attitudes towards risk. Could also be different for the same person at different wealth levels.

- attitudes toward risk
- measures of risk
- examples

**Definition 8.** A cumulative distribution function (CDF) $F : \mathbb{R} \to [0, 1]$ satisfies:

- $x \geq y$ implies $F(x) \geq F(y)$ (nondecreasing)
- $\lim_{y \downarrow x} F(y) = F(x)$ (right-continuous)
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

Notation:

- $\mu_F$ denotes the mean (expected value) of $F$, i.e., $\mu_F = \int x dF(x)$.
- $\delta_x$ is the degenerate distribution function at $x$, i.e., $\delta_x$ yields $x$.

$$\delta_x(z) = \begin{cases} 0 & \text{if } z < x \\ 1 & \text{if } z \geq x \end{cases},$$

which is essentially a way of viewing constants as distributions.

The space of all distribution functions is convex and one can define preferences over it. The expected utility is the integral of $u$ with respect to $F$:

$$\int u(x) dF(x) = \int u dF$$

A utility function $U$ on distributions, defined as $U(F) = \int u dF$ for some continuous index $u : \mathbb{R} \to \mathbb{R}$ over wealth such that

$$F \succsim G \iff \int u dF \geq \int u dG$$

We always think of $u$ (the utility index) as an increasing function.

**Definition 9** (Fair bet)**.** A bet is actuarially fair if it has expected value of zero.

How much would a decision-maker place on a bet that yields $ax$ with probability $p$ and a loss of $x$ with probability $1-p$? Consider the problem of finding an optimal $x$.

$$\max_x p \cdot u(w+ax) + (1-p)u(w-x)$$

where $w$ is the initial wealth level. F.O.C. w.r.t. $x$: $0 = pu'(w+ax)\cdot a - (1-p)u'(w-x) \iff \frac{pa}{1-p} = \frac{u'(w-x)}{u'(w+ax)}$. When the bet is fair, we have $a = \frac{1-p}{p}$, then

$$1 = \frac{u'(w-x)}{u'(w+ax)} \Rightarrow u'(w+ax) = u'(w-x).$$

Suppose $u$ is increasing and strictly concave. Then, $w-x = w+ax \Rightarrow x = 0$. Hence, this tells us that a decision-maker with a concave utility function will not bet any positive amount on a fair bet.

<u>An insurance problem</u>: An individual faces a potential accident with loss $L$ with probability $\pi$ and no loss with $1-\pi$.

**Definition 10** (Insurance Contract)**.** An insurance contract establishes a premium $p$ and reimburses $Z$ if and only if a loss occurs.

What is the expected profit of the insurance company? $p - \pi Z + (1-\pi)\cdot 0 = p - \pi Z$. Insurance will be a fair gamble when $p = \pi Z$. The individual will or will not buy insurance depending on their utility function. Expected utility with insurance:

$$\pi u(w-L-p+Z) + (1-\pi)u(w-p) \quad (1)$$

Expected utility without insurance:

$$\pi u(w-L) + (1-\pi)u(w) \quad (2)$$

Note that when $Z = L$, (1) becomes $u(w-p)$. The decision-maker will buy insurance if and only if (1)$\geq$(2), which would depend on $u, w, \pi, L, p, Z$. The problem becomes more complicated with information asymmetry (contract theory).

### 1.3.3 Concepts Related to Risk

#### Risk and Risk Aversion

Given some $F$, let $\delta_{\mu_F}$ be the distribution that yields the expected value of $F$ for sure.

**Definition 11** (Risk averse)**.** The preference relation $\succsim$ is risk averse if for all CDFs, $\delta_{\mu_F} \succsim F$, is risk-loving if $\delta_{\mu_F} \precsim F$, and is risk-neutral if $\delta_{\mu_F} \sim F$.

Note that the above definition does not depend on a specific utility representation.

---

**Example 5.** Let $\succsim$ be a preference relation on $\Delta(\mathbb{R})$ (the space of CDFs on $\mathbb{R}$). Consider the following utility

$$U(F) = \begin{cases} x & \text{if } F = \delta_x, x \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

This decision-maker is not risk averse, as $x$ can be negative. That is, for some $F$ with $\mu_F = -100$, $U(\delta_{\mu_F}) = -100$, $U(F) = 0$, and thus $U(F) > U(\delta_{\mu_F})$.

---

**Certainty Equivalent (CE)**

**Definition 12.** Given a strictly increasing and continuous von Neumann-Morgenstern utility index $u(\cdot)$ over wealth, the certainty equivalent of $F$ denoted $C(F, u)$ is defined by

$$u(C(F, u)) = \int u(x) dF(x).$$

Note that $C(F, u)$ is the amount such that $\delta_{C(F,u)} \sim F$. Unlike risk aversion, the certainty equivalent assumes a given utility representation. CE is related to risk aversion.

**Risk Premium**

**Definition 13.** Given a strictly increasing and continuous von Neumann-Morgenstern utility index on wealth, the risk premium is

$$r(F, u) = \mu_F - C(F, u)$$

Suppose that $\succsim$ satisfies all the axioms so that it has an expected utility representation. Then, risk aversion is completely characterized by the concavity of the utility index and a nonnegative risk premium.

**Proposition 1.** Suppose $\succsim$ has an expected utility representation and $u(\cdot)$ is the von Neumann-Morgenstern utility index. The following statements are equivalents:

i) $\succsim$ is risk averse

ii) $u$ is concave

iii) $r(F, u) \geq 0, \forall F$.

A quick aside: Jensen's inequality. A function $g$ is concave if and only if

$$\int g(x)dF(x) \leq g\left(\int xdF(x)\right), \text{ or } \mathbb{E}[g(X)] \leq g(\mathbb{E}[X]).$$

Note that when $u(\cdot)$ is increasing and $C(F,u) \leq \mu_F$, then $u(C(F,u)) \leq u(\mu_F)$.

*Proof.*

i)⇒ii) Suppose $\succsim$ is risk averse. By definition, $\delta_{\mu_F} \succsim F, \forall F \in \Delta(\mathbb{R})$. Fix $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$. Define a random variable $X$ such that

$$\mathbb{P}(X = x) = \alpha, \mathbb{P}(X = y) = 1 - \alpha.$$

Let $F^{\alpha}_{x,y}$ denote the CDF of $X$. By risk aversion, $\delta_{\mu_{F^{\alpha}_{x,y}}} \succsim F^{\alpha}_{x,y}$. Then, we have

$$u(\mu_{F^{\alpha}_{x,y}}) \geq \int u(z)dF^{\alpha}_{x,y}(z) \iff u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y).$$

Hence, $u(\cdot)$ is concave since $x, y, \alpha$ were arbitrary.

ii)⇒iii) Let $u$ be concave and $X$ a random variable with CDF $F$. By Jensen's inequality, we have

$$u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)] \iff u(\mu_F) \geq \int udF(x) = u(C(F,u)).$$

Since $u(\cdot)$ is increasing, $\mu_F \geq C(F,u) \Rightarrow r(F,u) = \mu_F - C(F,u) \geq 0, \forall F$.

iii)⇒i) Let $r(F,u) \geq 0, \forall F$. Then, $\mu_F \geq C(F,u) \iff u(\mu_F) \geq u(C(F,u)) = \int u(x)dF(x)$. Hence, $\delta_{\mu_F} \succsim F, \forall F \in \Delta(\mathbb{R})$.

From the above, we have equivalence of the three statements. □

Can we compare attitudes toward risk? ⇝ a concept of "relative risk aversion"

**Definition 14.** Given two preference relations $\succsim_1$, $\succsim_2$, we say $\succsim_1$ is more risk-averse than $\succsim_2$ if and only if

$$F \succsim_1 \delta_x \Rightarrow F \succsim_2 \delta_x, \forall F \in \Delta(X), x \in X.$$

Note that this definition does not assume anything about the preference relation.

**Proposition 2.** Suppose $\succsim_1$, $\succsim_2$ are such that they can be represented by a vNM utility indices $u_1$, $u_2$ ($u_i$: increasing in wealth, continuous). Then, the following are equivalent:

(a) $\succsim_1$ is more risk averse than $\succsim_2$

(b) $u_1 = \phi \circ u_2$ for some strictly increasing and concave $\phi : \mathbb{R} \to \mathbb{R}$

(c) $C(F, u_1) \leq C(F, u_2), \forall F$

(d) $r(F, u_1) \geq r(F, u_2), \forall F$

*Proof.* I employ a cyclical argument in the following order chosen for convenience: (a) ⇒ (c) ⇒ (d) ⇒ (b) ⇒ (a).

1. (a)⇒(c)
   Suppose $\succsim_1$ is more risk averse than $\succsim_2$. Consider an arbitrary distribution $F \in \Delta(\mathbb{R})$. By definition of a certainty equivalent, we have $\delta_{C(F,u_1)} \sim_1 F$ and $\delta_{C(F,u_2)} \sim_2 F$. Then, since $\succsim_1$ is more risk averse than $\succsim_2$ and $F \succsim_1 \delta_{C(F,u_1)}$, we have $F \succsim_2 \delta_{C(F,u_2)}$.

   By transitivity, $\delta_{C(F,u_2)} \succsim_2 \delta_{C(F,u_1)} \iff u_2(C(F,u_2)) \geq u_2(C(F,u_1))$. Since $u_2$ is a von Neumann-Morgenstern utility index (and thus increasing), we have $C(F,u_2) \geq C(F,u_1)$ as desired.

2. (c)⇒(d)
   Suppose $C(F, u_1) \leq C(F, u_2), \forall F$. Define $\mu_F := \int x dF(x)$, i.e., the expectation of $F$. Then,

$$
\begin{aligned}
&\iff -C(F,u_1) \geq -C(F,u_2) \\
&\iff \mu_F - C(F,u_1) \geq \mu_F - C(F,u_2) \\
&\iff r(F,u_1) \geq r(F,u_2), \forall F,
\end{aligned}
$$

   which is the desired. (In fact, this part of the proof is essentially bidirectional.)

3. (d)⇒(b)
   Suppose $r(F,u_1) \geq r(F,u_2), \forall F$. Then, $\mu_F - C(F,u_1) \geq \mu_F - C(F,u_2) \iff C(F,u_1) \leq C(F,u_2)$. Since $u_1$, $u_2$ are increasing von Neumann-Morgenstern indices, there exists an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $u_1 = \phi \circ u_2$. It suffices to check that $\phi$ is concave.

   Since $u_1$ is increasing, we have

$$
u_1(C(F,u_1)) \leq u_1(C(F,u_2)) \Rightarrow \int u_1(x) dF(x) \leq \phi(u_2(C(F,u_2)))
$$

   Plugging in $u_1 = \phi \circ u_2$ yields

$$
\int \phi(u_2(x)) dF(x) \leq \phi\left(\int u_2(x) dF(x)\right).
$$

Let $y := u_2(x)$. An appropriate change of measure from $F$ to $G$, where $F$ is absolutely continuous with respect to $G$, yields

$$\int \phi(y)dG(y) \leq \phi\left(\int ydG(y)\right),$$

which is Jensen's inequality. Thus, $\phi$ is concave.

4. (b)⇒(a)
Suppose $u_1 = \phi \circ u_2$ for some strictly increasing and concave $\phi : \mathbb{R} \to \mathbb{R}$. Consider a distribution $F \in \Delta(\mathbb{R})$ such that $F \succsim_1 \delta_x$ for some $x \in \mathbb{R}$. This is equivalent to

$$\int u_1(x)dF(x) \geq u_1(x) \iff \int (\phi \circ u_2)(x)dF(x) \geq (\phi \circ u_2)(x).$$

Since $\phi$ is concave, Jensen's inequality yields

$$\phi\left(\int u_2(x)dF(x)\right) \geq \int \phi(u_2(x))dF(x) \geq \phi(u_2(x)).$$

Because $\phi$ is strictly increasing,

$$\int u_2(x)dF(x) \geq u_2(x) \iff F \succsim_2 \delta_x,$$

which is the desired.

From the above, I have established that the four statements are equivalent. □

How do we measure risk aversion? Since the concavity of the utility function captures risk aversion, a guess would be that $u''$ is a natural candidate. However, $u''$ is not a good measure because vNM utilities are only unique up to affine transformation. For example, $u_1(x) = 100x^3$ and $u_2(x) = x^3$ represent the same preference but there is a big difference in the second derivative.

**Definition 15** (Arrow-Pratt measure of risk aversion)**.** Suppose $\succsim$ is a preference relation represented by a twice-differentiable vNM utility function $u : \mathbb{R} \to \mathbb{R}$. The Arrow-Pratta measure of absolute risk aversion is defined by

$$\lambda(x) = -\frac{u''(x)}{u'(x)}.$$

**Proposition 3.** Suppose $\succsim_1, \succsim_2$ are preference relations represented by twice-differentiable vNM utility indices $u_1$ and $u_2$. Then, $\succsim_1$ is more risk averse than $\succsim_2$ if $\lambda_1(x) \geq \lambda_2(x), \forall x \in \mathbb{R}$.

*Proof.* From the previous proposition, we know that there exists a strictly increasing and concave function $\phi : \mathbb{R} \to \mathbb{R}$ such that $u_1 = \phi \circ u_2$. Note that this implies $\phi' > 0$ and $\phi'' \leq 0$. Then,

$$\begin{aligned} u_1'(x) &= \phi'(u_2(x)) \cdot u_2'(x) \\ u_1''(x) &= \phi''(u_2(x)) \cdot (u_2'(x))^2 + \phi'(u_2(x)) \cdot u_2''(x) \\ \Rightarrow -\lambda_1(x) &= \frac{\phi''(u_2(x)) \cdot (u_2'(x))^2 + \phi'(u_2(x)) \cdot u_2''(x)}{\phi'(u_2(x)) \cdot u_2'(x)} = \frac{\phi'' \cdot (u_2'(x))^2}{\phi' \cdot u_2'} + \frac{u_2''}{u_2'} \\ \Rightarrow \lambda_1(x) &= \lambda_2(x) - \frac{\phi'' \cdot (u_2'(x))^2}{\phi' \cdot u_2'} \geq \lambda_2(x), \forall x \in \mathbb{R}. \end{aligned}$$

□

Now we shift gears and talk about orders on distributions.

**Definition 16** (First-order stochastic dominance)**.** For $F, G \in \Delta(\mathbb{R})$, $F \succsim_{FOSD} G$ ($F$ first-order stochastically dominates $G$) if $\int u dF \geq \int u dG$ for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$.

*Remark.* $\succsim_{FOSD}$ is a binary relation on the set of distributions. It is, in fact, a partial order (not complete).

**Proposition 4** (Alternative characterization of FOSD)**.** $F \succsim_{FOSD} G \iff F(x) \leq G(x), \forall x \in \mathbb{R}$.

*Proof.*

($\Leftarrow$) Suppose $F(x) \leq G(x), \forall x \in \mathbb{R}$. Define $\varphi(x) := F(x) - G(x), \forall x \in \mathbb{R}$. Then, we have $\varphi(x) \leq 0, \forall x \in \mathbb{R}$. Since $F, G$ are CDFs, we should have the following:

$$\lim_{x \to -\infty} F(x) = 0, \lim_{x \to -\infty} G(x) = 0, \lim_{x \to \infty} F(x) = 1, \lim_{x \to \infty} G(x) = 1.$$

This implies

$$\lim_{x \to -\infty} \varphi(x) = \lim_{x \to \infty} \varphi(x) = 0.$$

Let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary nondecreasing function. Since $u(x)$ is nondecreasing, we have $u'(x) \geq 0, \forall x \in \mathbb{R}$ (assuming differentiability). Then, integration by parts gives

$$\int u(x) d\varphi(x) = u(x)\varphi(x)|_{-\infty}^{\infty} - \int u'(x)\varphi(x)dx = -\int u'(x)\varphi(x)dx \geq 0.$$

The above implies

$$\int u(x) dF(x) \geq \int u(x) dG(x) \iff F \succsim_{FOSD} G,$$

as desired.

($\Rightarrow$) Suppose $F \succsim_{FOSD} G$. Using integration by parts,

$$\begin{aligned}\int_a^b u(x)dF(x) &= u(x)F(x)\Big|_a^b - \int_a^b u'(x)F(x)dx \\ &= u(b)\cdot 1 - u(a)\cdot 0 - \int_a^b u'(x)F(x)dx\end{aligned}$$

$$\begin{aligned}\int_a^b u(x)dG(x) &= u(x)G(x)\Big|_a^b - \int_a^b u'(x)G(x)dx \\ &= u(b)\cdot 1 - u(a)\cdot 0 - \int_a^b u'(x)G(x)dx\end{aligned}$$

$$\begin{aligned}\Rightarrow -\int_a^b u'(x)F(x)dx &\geq -\int_a^b u'(x)G(x)dx \\ \Rightarrow \int_a^b u'(x)[G(x)-F(x)]dx &\geq 0, \forall u:\mathbb{R}\to\mathbb{R} \text{ increasing} \\ &\Rightarrow G(x) \geq F(x), \forall x \in \mathbb{R}\end{aligned}$$

as desired.

□

**Definition 17** (Second order stochastic dominance)**.** For $F, G \in \Delta(\mathbb{R})$, $F \succsim_{SOSD} G$ ($F$ second-order stochastically dominates $G$) if $\int udF \geq \int udG$ for every nondecreasing, concave function $u : \mathbb{R} \to \mathbb{R}$.

*Remark.* If $F \succsim_{SOSD} G$, everyone who prefers more money and is risk-averse prefers $F$ over $G$.

**Proposition 5.**

$$F \succsim_{SOSD} G \iff \int_{-\infty}^x F(t)dt \leq \int_{-\infty}^x G(t)dt, \forall x \in \mathbb{R}.$$

### 1.3.4 Anscombe-Aumann Structure

The vNM formulation allows only for objective uncertainty. In a general framework, we also have *subjective* uncertainty. In most cases, probabilities about certain outcomes are not given but are in the mind of the decision maker. A decision maker's preference relation should reveal her "beliefs" (distribution over events) as well as her preference relation over "consequences."

**Anscombe-Aumann Structure**[8]

This framework allows both for subjective and objective uncertainty. We have the following ingre-

8. This is an extension of Savage's model, which only allows for subjective uncertainty.

dients:

- $X = \{x_1, \dots, x_n\}$: a finite set of outcomes
- $\Omega = \{s_1, \dots, s_m\}$: a finite set of states of the world
- a set of "acts" (new!)

**Definition 18** (Anscombe-Aumann acts)**.** An Anscombe-Aumann act is a function $h : \Omega \to \Delta(X)$ (i.e., from states to gambles).

*Remark.* Interpretation: elements of $\Delta(X)$ are bets on an objective roulette. Outcome probabilities are objective, which means all decision makers agree on them (e.g., flipping a coin). On the other hand, beliefs abut the state are subjective. That is, each person has their own probability distribution over $\Omega$ (e.g., I might assign high probability to $s_1$, whereas someone might assign 0). Once a state is realized, we have a common gamble that we all agree upon.

A story for intuition: A state of the world $s$ represents the event that a specific horse named "$s$" wins the race among all possible horses in $\Omega$. Each decision maker subjectively assesses each horse's strength (chances of winning the race). Once a particular "horse" wins, a particular roulette is spun based on which "horse" won.

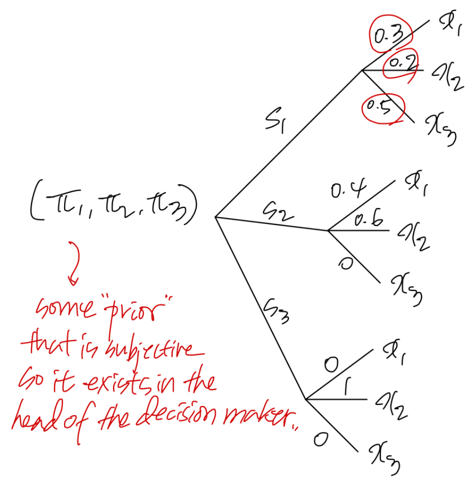
The set of acts is denoted by $H$. There are three different ways to represent $H$.

**Example 6.** Consider the sets $\Omega = \{s_1, s_2, s_3\}$ and $X = \{x_1, x_2, x_3\}$.

1. The original mathematical definition. $h : \Omega \to \Delta(X)$, $H = \Delta X^\Omega$

$$h(s_1) = (0.3, 0.2, 0.5)$$
$$h(s_2) = (0.4, 0.6, 0)$$
$$h(s_3) = (0, 1, 0)$$

2. A compound lottery.




$\rightsquigarrow (\pi_1 \circ s_1, \pi_2 \circ s_2, \pi_3 \circ s_3) \rightsquigarrow (p_1 \circ x_1, p_2 \circ x_2, p_3 \circ x_3)$. Note that the probabilities of the induced simple gamble $p_i$ combine subjective prior probabilities $\pi_i$ and the objective probabilities given by $h$.

3. A matrix

$$h = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.6 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

---

In the A-A model, the decision maker has preferences over a continuous $H$ (i.e., preferences over functions).

(Detour)
Let $\Pi$ denote some convex subset of $\mathbb{R}^n$ (it need not be; the actual results are more general).

A convex set: if $x, y \in \Pi$, then $\alpha x + (1-\alpha)y \in \Pi, \forall \alpha \in [0,1]$.

On convex consumption sets, we can use a weaker version of the continuity axiom.

- Archimedean axiom: a binary relation on $\Pi$ is Archimedean if $\forall \pi, \rho, \sigma \in \Pi$,

$$\pi \succ \rho \succ \sigma \Rightarrow \begin{cases} \exists \alpha \in (0,1) \text{ s.t. } \alpha\pi + (1-\alpha)\sigma \succ \rho \\ \exists \beta \in (0,1) \text{ s.t. } \rho \succ \beta\pi + (1-\beta)\sigma. \end{cases}$$

Exercise:

- $\succsim$: continuous $\Rightarrow$ $\succsim$: Archimedean
- An example of $\succsim$ that is Archimedean but not continuous.

*Proof.* Let $\succsim$ on $\Pi \subseteq \mathbb{R}^n$ be continuous. Consider $\pi, \rho, \sigma \in \Pi$ with $\pi \succ \rho \succ \sigma$.

As a result of continuity, we know that the line segment joining $\pi$ and $\sigma$, call it $S$, has a nonempty intersection with $\sim(\rho)$ (Problem 1 of PS2). Note that $S$ is closed and so is $\sim(\rho)$, which implies that $S \cap \sim(\rho)$ is also closed.

Then, we can write $S \cap \sim(\rho) = \{t \in [\underline{t}, \bar{t}] : t\pi + (1-t)\rho\}$, where $\underline{t} > 0$, $\bar{t} < 1$. We can choose $\alpha \in (\bar{t}, 1), \beta \in (0, \underline{t})$, which will yield

$$\alpha\pi + (1-\alpha)\sigma \in \succ(\rho), \quad \beta\pi + (1-\beta)\sigma \in \succ(\rho).$$

□

- Independence axiom: a binary relation on $\Pi$ satisfies independence if $\forall x, y, z \in \Pi, \alpha \in (0,1)$,

$$x \succsim y \iff \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

Exercise: Find an example of a preference relation that violates independence.

**Theorem 20** (Mixture Space Theorem (Hernstein and Milner))**.** *A binary relation on $\succsim$ on a convex subset $\Pi \subseteq \mathbb{R}^n$ is complete, transitive, independent and Archimedean if and only if there exists an affine function $u : \Pi \to \mathbb{R}$ representing $\succsim$ such that $\pi \succsim \rho \iff u(\pi) \geq u(\rho)$.*

*Moreover, if $u : \Pi \to \mathbb{R}$ is an affine representation of $\succsim$, then $\tilde{u} : \Pi \to \mathbb{R}$ is an affine representation of $\succsim$ if and only if there exist real numbers $a > 0, b$ such that*

$$\tilde{u}(\pi) = au(\pi) + b, \forall \pi \in \Pi.$$

Comments: Let $\Pi = \Delta(X)$, which is a convex set. Then, the MST implies the vNM representation theorem.

**Theorem 21.** *The preference relation $\succsim$ on $H$ is complete, transitive, independent and Archimedean* *if and only if there exist vNM indices* $u_1, \ldots, u_n : X \to \mathbb{R}$ *such that*

$$u(h) = \sum_s \sum_x h_s(x) u_s(x)$$

*is a utility representation of* $\succsim$.

*Proof.* The proof follows immediately from applying MST to the set $H$. □

But this is not good enough! Ideally, we would like a representation of the following form:

$$u(h) = \sum_s \mu(s) \left[ \sum_x h_s(x) u_s(x) \right].$$

That is, we would like to identify both $\mu$ (the beliefs) and $u_s$ (the preference on $X$) from the data. This is <u>not</u> possible. Suppose we had $\mu, u_s$ representing $\succsim$ on $H$. Then, for any $\mu' \in \Delta(\Omega)$ such that $\mu'(s) > 0, \forall s \in \Omega, \exists u'_1, \ldots, u'_n$ such that

$$u(h) = \sum_s \mu'(s) \sum_x h_s(x) u'_s(x),$$

i.e., $\mu'(s) u'_s = \mu(s) u_s$ with $\mu' \neq \mu, u'_s \neq u_s$. (Much like a degree of freedom problem.)

Conclusion: one cannot identify probabilities using state-dependent utility.

A-A framework provides a state-independent representation.

$H \in \Delta(X)^\Omega$: the set of all acts. We can view $X$ and $\Delta(X)$ as acts.

$$\Delta X = H_c = \{f \in H : f(s) = f(s'), \forall s, s' \in \Omega\},$$

i.e., a *constant* act. The same distribution regardless of state.

$$X = \{f \in H : f(s) = f(s'), \forall s, s' \in \Omega, f(s) = \delta_x, x \in X\},$$

i.e., a constant act that also yields a Dirac delta distribution. The same distribution regardless of state *and* that distribution is a degenerate one.

With this language, we can proceed to formally define state-independent utility, which relies on first defining null states (the states that do not matter).

To define null states, we need the following:
Given an act $h \in \Delta X^\Omega$ and a state $s$ and a lottery $\pi \in \Delta X$, define a new act

$$(h_{-s}, \pi) : \Pi \to \Delta X, (h_{-s}, \pi) = (h_1, \ldots, h_{s-1}, \pi, h_{s+1}, \ldots)$$

i.e., $(h_{-s}, \pi) = \begin{cases} \pi & \text{if } t = s \\ h(t) & \text{o.w.} \end{cases}$

**Definition 19** (Null state)**.** A state $s \in \Omega$ is null if, for all $h \in \Delta X^{\Omega}$ and all $\pi, \rho \in \Delta X$, we have $(h_{-s}, \pi) \sim (h_{-s}, \rho)$. A state $s \in \Omega$ is non-null if it is not null, i.e., $\exists h \in \Delta X^{\Omega}$ and $\pi, \rho \in \Delta X$ such that $(h_{-s}, \pi) \succ (h_{-s}, \rho)$.

**Definition 20** (State independent preference relation)**.** The binary relation $\succsim$ on $H$ is state-independent if for all non-null states $s \in \Omega$, for all acts $h, g \in H$ and for all lotteries $\pi, \rho \in \Delta X$,

$$(h_{-s}, \pi) \succsim (h_{-s}, \rho) \Rightarrow (g_{-t}, \pi) \succsim (g_{-t}, \rho).$$

The ranking of lotteries $\pi, \rho$ does not depedent on the state. And this hs to be true for all states that the decision maker cares about. This definition implies that $u_s(\cdot) = u(\cdot)$, that is the utility index over consequences does not depend on the state: $u_s(x) \geq u_s(x'), u_t(x) \geq u_t(x')$.

**Example 7.** Consider the following setting. $\Omega = \{\text{rain}, \text{shine}\}, X = \{\text{umbrella}, \text{sunglasses}\}$

$$(\delta_u, \delta_s) \succ (\delta_u, \delta_u), (\delta_u, \delta_s) \succ (\delta_s, \delta_s)$$

This preference is not state-independent, as we have $\delta_s \succ \delta_u$ from the first and $\delta_u \succ \delta_s$ from the second.

**Theorem 22** (Expected Utility Theorem of Anscombe-Aumann)**.** *A preference relation on $H$ is independent, Archimedean and state-independent if and only if there exists a vNM utility index $u : X \to \mathbb{R}$ and a probability distribution $\mu$ on $\Omega$ ($\mu \in \Delta\Omega$) such that*

$$u(h) = \sum_s \underbrace{\mu(s)}_{\textit{subjective uncertainty}} \sum_x \underbrace{h_s(x)}_{\textit{objective uncertainty}} u(x)$$

*Remark.* To find out the total probability of outcome $x$,

$$\underbrace{\sum_{x \in X} u(x)}_{\text{utility over all possible } x} \underbrace{\sum_{s \in S} \mu(s) h_s(x)}_{\text{probability of } x}$$

Preference here identifies the utility index $u : X \to \mathbb{R}$ and a probability measure over states $\mu \in \Delta\Omega$. VNM theorem identifies only a utility index.

# 2 Producer Theory

Producers purchase inputs and turn them into outputs (according to a production function).

Properties of production (possibility) set $Y \subseteq \mathbb{R}^n$:

1. No free lunch: $Y \cap \mathbb{R}^n_+ \subseteq \{0_n\}$ (cannot have all outputs without any input)
2. Possibility of inaction: $0_n \in Y$
3. Free disposal: $y \in Y$ implies $y' \in Y$ for all $y' \leq y$ (can make anything strictly less with same input)
4. Irreversibility: if $y \in Y$ and $y \neq 0_n$ then $-y \in Y$ (cannot invert input and output)
5. Nonincreasing returns to scale: If $y \in Y$, then $\alpha y \in Y, \forall \alpha \in [0,1]$
6. Nondecreasing returns to scale: If $y \in Y$, then $\alpha y \in Y, \forall \alpha \geq 1$
7. Constant returns to scale: If $y \in Y$, then $\alpha y \in Y, \forall \alpha \geq 0$
8. Additivity: If $y, y' \in Y$, then $y + y' \in Y$
9. Convexity: $Y$ is convex, i.e., $\alpha y + (1-\alpha)y' \in Y, \forall y, y' \in Y$
10. $Y$ is a convex cone if for any $y, y' \in Y$ and $\alpha, \beta \geq 0$, $\alpha y + \beta y' \in Y$.

Some properties:

- $Y$: additive, nonincreasing returns to scale $\iff$ $Y$: convex cone
- For all $Y \subseteq \mathbb{R}^n$ with $0_n \in Y$, there exists a convex $Y' \subseteq \mathbb{R}^{n+1}$ such that satisfies constant returns and

$$Y = \{y \in \mathbb{R}^n : (y, -1) \in Y'\}$$

*Proof of first statement.*

($\Rightarrow$) Suppose $Y$ is additive and satisfies nonincreasing returns to scale. Consider $y, y' \in Y$. Then, by nonincreasing returns to scale, we have $\alpha_1 y \in Y$, $\forall \alpha_1 \in [0,1]$.

Given $\alpha \geq 0$, we can find $\{a_n\} \subseteq [0,1]$ such that $\alpha_1 + \cdots + \alpha_n = \alpha$. Therefore, for any $\alpha \geq 0$, we have $\alpha y \in Y$ by additivity. A similar argument applies to $\beta y' \in Y$ for any $\beta \geq 0$. Again, by additivity, $\alpha y + \beta y' \in Y$.

Since $y, y', \alpha, \beta$ were arbitrary, $Y$ is a convex cone.

($\Leftarrow$) Suppose $Y$ is a convex cone. Consider $y, y' \in Y$. Then, $\alpha y + \beta y' \in Y, \forall \alpha, \beta \geq 0$. We have $y + y' \in Y$ by choosing $\alpha = \beta = 1$. Hence, $Y$ is additive.

Moreover, we can choose $\beta = 0$ and any $\alpha \in [0,1]$, which yields $\alpha y \in Y, \forall \alpha \in [0,1]$. Hence, $Y$ is of nonincreasing returns to scale.

□

*Proof of second statement.* (Adapted from Mas-Colell, Whinston, and Green (1995))

Define $Y' := \{y' \in \mathbb{R}^{n+1} : y' = (\alpha y, -\alpha), y \in Y, \alpha \geq 0\}$.

First, $Y'$ satisfies constant returns to scale. Let $y' \in Y'$. Then, there exists $y \in Y$ and $\alpha \geq 0$ such that $y' = (\alpha y, -\alpha)$. Then, for any $\beta \geq 0$, $\beta y' = (\beta\alpha y, -\beta\alpha)$, and therefore $\beta y' \in Y'$.

Now, consider an arbitrary $y \in Y$. Then, there exists $y' \in Y'$ such that $y' = (y, -1)$ by construction. Hence, the second part of the statement is also true. □

Let $y \in \mathbb{R}^m$ denote outputs and $x \in \mathbb{R}^n$ represent inputs. If the two are related by a function $f : \mathbb{R}^n_+ \to \mathbb{R}^m_+$, this is called a *production function*, $y = f(x)$. The corresponding production possibility set is:

$$Y = \{(-x, y) \in \mathbb{R}^n_- \times \mathbb{R}^m_+ : y \leq f(x)\}$$

When $m = 1$,

- $Y$: constant returns to scale $\iff$ $f$: homogeneous of degree 1, i.e., $f(\alpha x) = \alpha f(x), \forall \alpha \geq 0$
- $Y$: convex $\iff$ $f$: concave

*Proof of first statement.*

($\Rightarrow$) Suppose $Y$ has constant returns to scale. Consider a vector of inputs $x$. Then, $(-x, f(x)) \in Y$ by definition as $f(x) \leq f(x)$.

Since $Y$ has constant returns to scale, $(-\alpha x, \alpha f(x)) \in Y$ for any $\alpha \geq 0$. Then, $\alpha f(x) \leq f(\alpha x)$. It now suffices to show that $\alpha f(x) \geq f(\alpha x)$.

Fix $\alpha > 0$ and consider the combination $(-\alpha x, f(\alpha x) \in Y$. Again, $Y$ having constant returns to scale yields

$$(-x, \frac{1}{\alpha} f(\alpha x)) \in Y \Rightarrow \frac{1}{\alpha} f(\alpha x) \leq f(x) \Rightarrow f(\alpha x) \leq \alpha f(x).$$

Note that the above will continue to hold as $\alpha \to 0$. Therefore, $f(\alpha x) = \alpha f(x), \forall \alpha \geq 0$, i.e., the production function is homogeneous of degree 1.

($\Leftarrow$) Suppose $f$ is homogeneous of degree 1. Then, $f(\alpha x) = \alpha f(x), \forall \alpha \geq 0$. Let $(-x, y) \in Y$. Then, for any $\alpha \geq 0$, $\alpha y \leq \alpha f(x) = f(\alpha x)$. Hence, $(-\alpha x, \alpha y) \in Y$, which implies $Y$ has constant returns to scale.

□

*Proof of second statement.*

($\Rightarrow$) Suppose $Y$ is convex. Consider two vectors of inputs $x, x'$.

Then, $(-x, f(x)), (-x', f(x')) \in Y$. By convexity of $Y$,

$$\begin{aligned}&(-(\alpha x + (1-\alpha)x'), \alpha f(x) + (1-\alpha)f(x')) \in Y \\ &\qquad \iff \alpha f(x) + (1-\alpha)f(x') \leq f(\alpha x + (1-\alpha)x'), \forall \alpha \in [0,1].\end{aligned}$$

Hence, $f$ is concave.

($\Leftarrow$) Suppose $f$ is concave. Then, for any two vectors of inputs $x, x'$, we have

$$\alpha f(x) + (1-\alpha)f(x') \leq f(\alpha x + (1-\alpha)x'), \forall \alpha \in [0,1].$$

For any $(-x, y), (-x', y') \in Y$, we have $y \leq f(x), y' \leq f(x')$ by definition. Then,

$$\begin{aligned}&\alpha y \leq \alpha f(x), \quad (1-\alpha)y' \leq (1-\alpha)f(x') \\ &\qquad \Rightarrow \alpha y + (1-\alpha)y' \leq \alpha f(x) + (1-\alpha)f(x') \leq f(\alpha x + (1-\alpha)x')\end{aligned}$$

Therefore, $(-(\alpha x + (1-\alpha)x'), \alpha f(x) + (1-\alpha)f(x')) \in Y$, i.e., $Y$ is convex.

□

**Definition 21** (Transformation function)**.** Given a production set $Y \subseteq \mathbb{R}^n$, the transformation function $F : Y \to \mathbb{R}$ is defined by

$$Y = \{y \in Y : F(y) \leq 0, F(y) = 0 \text{ if and only if } y \text{ is on the boundary of } Y\}$$

The transformation frontier is $\{y \in \mathbb{R}^n : F(y) = 0\}$.

**Definition 22** (Marginal rate of transformation)**.** Given a differentiable transformation $F$ and a point on its frontier $y$, the marginal rate of transformation for goods $i$ and $j$ is given

$$\text{MRT}_{i,j} = \frac{\frac{\partial F(y)}{\partial y_i}}{\frac{\partial F(y)}{\partial y_j}}$$

Since we have $F(y) = 0$,

$$\frac{\partial F(y)}{\partial y_i} dy_i + \frac{\partial F(y)}{\partial y_j} dy_j = 0$$

and therefore

$$-\frac{dy_j}{dy_i} = \frac{\frac{\partial F(y)}{\partial y_i}}{\frac{\partial F(y)}{\partial y_j}}$$

Firms: black box that turns into outputs (basically view it as its production function). Assume price-taker in both input and output. Objective: maximize profits/minimize costs. (Must assume decreasing returns to scale; cannot rule out producing infinitely many, i.e., the problem is not well-defined.)

We will assume that the production function $f$ is continuous, strictly increasing and strictly quasiconcave on $\mathbb{R}^n_+$ and $f(0) = 0$.

**Definition 23** (Isoquant)**.** An isoquant is a collection of input combinations which keep output fixed, i.e.,

$$Q(y) = \{x \in \mathbb{R}^n_+ : f(x) = y\}$$

(like an indifference curve for consumers)

Marginal rate of technical substitution (MRTS): for any two inputs $i, j$,

$$\text{MRTS}_{i,j}(x_1, x_2, \ldots, x_n) = \frac{MP_i(x)}{MP_j(x)}$$

## 2.1 Cost minimization

Firms seek to maximize profits, i.e., produce a given level of output $y$ in the cheapest possible way.

The cost function:

$$c(w, y) = \min_{x \in \mathbb{R}^n_+} w \cdot x \text{ s.t. } f(x) \geq y$$

where $w = (w_1, \ldots, w_n) \gg 0$, vector of input prices and $y \geq 0$. FOCs of this problem are identical to those of expenditure minimization.

$$\text{MRTS}_{i,j}(x^*) = \frac{w_i}{w_j}, f(x^*) = y$$

The solution to the above minimization problem $x^*(w, y)$ is called the *conditional demand input vector*. Conditional as we condition on a given output level $y$ (i.e., firm has already decided to produce $y$).

**Theorem 23** (Properties of cost functions)**.** *If $f$ is strictly increasing, strictly quasiconcave and continuous, then $c(w, y)$:*

1. *is zero when* $y = 0$
2. *is continuous in its domain*
3. *for all* $w \gg 0$*, is strictly increasing and unbounded above in* $y$
4. *is increasing in* $w$
5. *is homogeneous of degree 1 in* $w$
6. *is concave in* $w$
7. *satisfies Shephard's lemma if* $c$ *is differentiable in* $w$*, i.e,*

$$\frac{\partial c(w^0, y^0)}{\partial w_i} = x_i(w^0, y^0)$$

Note that we have integrability for producer theory, which is essentially identical to consumer theory. Integrating the demand retrieves the technology of the firm.

**Theorem 24** (Properties of conditional input demands)**.** *If* $f$ *is continuous, strictly increasing and strictly quasiconcave and* $c(w, y)$ *is twice continuously differentiable (*$c \in C^2$*), then*

1. $x(w, y)$ *is homogeneous of degree 0*
2. *the substitution matrix* $(\partial x_i / \partial w_j)_{n \times n}$ *is symmetric and negative semidefinite*

## 2.2 Profit Maximization

Competitive firms seek to maximize profits (assuming a single product firm):

$$\max_{y, x \in \mathbb{R}^n_+} p \cdot y - w \cdot x \text{ s.t. } f(x) \geq y, w \in \mathbb{R}^n_{++}, p \geq 0$$

where $w, p$ are exogenously given. Since $f(\cdot)$ is strictly increasing so the constraint $f(x) \geq y$ binds. Thus,

$$\max_{x \in \mathbb{R}^n_+} p \cdot f(x) - w \cdot x$$

If the solution $x^* \gg 0$, it is an interior solution. The FOCs are

$$p \frac{\partial f(x^*)}{\partial x_i} - w_i = 0$$

If $x^*$ is an optimal combination of inputs, $y^* = f(x^*)$ and thus

$$\text{MRTS}_{i,j} = \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}} = \frac{w_i}{w_j},$$

which is also true if we are minimizing costs. Additionally, from the FOC,

$$p = \frac{w_i}{MP_i} \Rightarrow pf_i(x^*) = pMP_i(x^*) = w_i$$

The **profit function** is the value function of the previous program at prices $(p, w)$:

$$\pi(p, w) = \max_{y,x \in \mathbb{R}^n_+} py - wx \text{ s.t. } f(x) \geq y$$

Note that $\pi$ is not necessarily well-defined (i.e., may not have a solution). (You can make infinite amount of the good, and reach an infinite profit.) It is generally well-defined for technologies that exhibit decreasing returns to scale. For constant and higher returns to scale, we cannot solve. For example, suppose the production function $f$ exhibits constant returns to scale, $f(10x) = 10y$. $\pi$ is either 0 (not feasible to produce anything) or $\infty$ (produce infinitely).

**Theorem 25** (Properties of profit functions)**.** *If $f$ is continuous, strictly increasing and strictly quasiconcave, then for $(p, w) \gg 0$ the profit function, whenever well-defined,*

1. *is increasing in $p$*
2. *is decreasing in $w$*
3. *is homogeneous of degree 1 in $(p, w)$*
4. *is convex in $(p, w)$*
5. *if differentiable in $(p, w) \gg 0$, then satisfies Hotelling's lemma:*

$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w)$$

$$-\frac{\partial \pi(p, w)}{\partial w_i} = x_i(p, w)$$

*where $y(p, w)$ is the output supply function and $x_i(p, w)$ is the input demand function (i.e., demand for labor).*

*Proof.*

1. Increasing in $p$:
   Consider $p' \gg p$. Let $(x, y)$ be a solution to

$$\max_{y,x \in \mathbb{R}^n_+} p \cdot y - w \cdot x$$

subject to $f(x) \geq y$, and $(x', y')$ be a solution to

$$\max_{y,x \in \mathbb{R}^n_+} p' \cdot y - w \cdot x$$

subject to $f(x) \geq y$. Note that $(x, y)$ is feasible for the second maximization, which implies

$$\pi(p', w) \geq p' \cdot y - w \cdot x > p \cdot y - w \cdot x = \pi(p, w)$$

as $\pi(p', w)$ is the maximum. Hence, $\pi(p', w) > \pi(p, w)$.

2. Decreasing in $w$:
   Consider $w' \ll w$. Let $(x, y)$ be a solution to

$$\max_{y,x \in \mathbb{R}^n_+} p \cdot y - w \cdot x$$

   subject to $f(x) \geq y$, and $(x', y')$ be a solution to

$$\max_{y,x \in \mathbb{R}^n_+} p \cdot y - w' \cdot x$$

   subject to $f(x) \geq y$. Note that $(x', y')$ is feasible for the first maximization. Then,

$$\pi(p, w') = p \cdot y' - w' \cdot x' < p \cdot y' - w \cdot x' \leq \pi(p, w),$$

   as $\pi(p, w)$ is the maximum. Hence, $\pi(p, w') < \pi(p, w)$.

3. Homogeneity of degree 1:
   Consider $t > 0$. Then,

$$\begin{aligned}
\pi(tp, tw) &= \arg\max_{f(x) \geq y} (tp) \cdot y - (tw) \cdot x \\
&= \arg\max_{f(x) \geq y} t(p \cdot y - w \cdot x) \\
&= t \cdot \arg\max_{f(x) \geq y} p \cdot y - w \cdot x \\
&= t \cdot \pi(p, w).
\end{aligned}$$

   Hence, $\pi(p, w)$ is homogeneous of degree 1 in $(p, w)$.

4. Convex in $(p, w)$
   Consider $(p^0, w^0), (p^1, w^1), t \in [0, 1]$. Define $\bar{p} := tp^0 + (1-t)p^1$ and $\bar{w} := tw^0 + (1-t)w^1$.

Let $(x, y)$ be a solution to

$$\max_{y,x\in\mathbb{R}^n_+} \bar{p} \cdot y - \bar{w} \cdot x$$

subject to $f(x) \geq y$. Then, we have

$$\begin{aligned}
\pi(\bar{p}, \bar{w}) &= \bar{p} \cdot y - \bar{w} \cdot x \\
&= (tp^0 + (1-t)p^1) \cdot y - (tw^0 + (1-t)w^1) \cdot x \\
&= t(p^0 \cdot y - w^0 \cdot x) + (1-t)(p^1 \cdot y - w^1 \cdot x) \\
&\leq t\pi(p^0, w^0) + (1-t)\pi(p^1, w^1)
\end{aligned}$$

as we have $f(x) \geq y$ (i.e., feasible) and $\pi(\cdot, \cdot)$ is the maximum given any arguments.

5. Hotelling's lemma:
   Let $x(p, w), y(p, w)$ be solutions to profit maximization, i.e.,

$$\pi(p, w) = p \cdot y(p, w) - w \cdot x(p, w).$$

By envelope theorem, we have

$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w) + p \cdot \underbrace{\frac{\partial y(p, w)}{\partial p}}_{=0} - w \cdot \underbrace{\frac{\partial x(p, w)}{\partial p}}_{=0} = y(p, w).$$

Similarly, by envelope theorem, we have

$$\frac{\partial \pi(p, w)}{\partial w} = p \cdot \underbrace{\frac{\partial y(p, w)}{\partial w}}_{=0} - x(p, w) - w \cdot \underbrace{\frac{\partial x(p, w)}{\partial w}}_{=0} = -x(p, w)$$

as desired.

□

Note that both input and output prices affect the profit-maximizing output choice $y$ and the corresponding input choice $x$. Now, $x_i(p, w)$ here is the unconditional input demand in contrast to the solution to the cost minimization problem. The supply function is:

$$y^*(p, w) = f(x^*(p, w))$$

The substitution matrix is a $(n+1)\times(n+1)$ matrix:

$$\sigma(p,w) = \begin{bmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial w_1} & \cdots & \frac{\partial y}{\partial w_n} \\ -\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial w_1} & \cdots & -\frac{\partial x_1}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial x_n}{\partial p} & -\frac{\partial x_n}{\partial w_1} & \cdots & -\frac{\partial x_n}{\partial w_n} \end{bmatrix}$$

**Theorem 26** (Properties of profit-maximizing choices)**.** *Suppose $f$ is continuous, strictly increasing and strictly quasiconcave and $\pi(p,w)$ is twice continuously differentiable, then*

1. *$y(p,w)$ and $x(p,w)$ are homogeneous of degree 0 in all prices*
2. *$\frac{\partial y}{\partial p}\geq 0$ (no infereior good), $\frac{\partial x_i}{\partial w_i}\leq 0$ (no inferior inputs)*
3. *The substitution matrix is symmetric and positive semidefinite because $\pi$ is convex.*

Multiproduct firms (follow MWG from here on): profit maximization using the representation of production possibilities as the transformation frontier ($F(y)=0$).

$$\max_{y\in Y} p\cdot y$$

or equivalently

$$\max_{y} p\cdot y \text{ s.t. } F(y)=0$$

FOC:

$$p_i = \lambda\frac{\partial F(y)}{\partial y_i}, \forall i \Rightarrow p = \lambda\nabla F(y)$$

The FOC can be rearranged to

$$\frac{1}{\lambda} = \frac{\partial F(y)/\partial y_i}{p_i}, \forall i$$

This implies

$$\text{MRT}_{i,j} = \frac{\frac{\partial F(y)}{\partial y_i}}{\frac{\partial F(y)}{\partial y_j}} = \frac{p_i}{p_j}, \forall i,j$$

That is, the marginal rate of transformation equals the price ratio.

Given a production set $Y\subseteq\mathbb{R}^n$, the supply correspondence $y^*:\mathbb{R}^n_{++}\to\mathbb{R}^n$ is defined as

$$y^*(p) = \arg\max_{y\in Y} p\cdot y$$

The profit function $\pi : \mathbb{R}^n_{++} \to \mathbb{R}$ given $Y \subseteq \mathbb{R}^n$ is defined as

$$\pi(p) = \max_{y \in Y} p \cdot y$$

The supply could be a set (hence correspondence), but the profit must be unique (or else it would not belong in the set). The value function is always unique! (If well-defined, of course.)

# 3 Walrasian Equilibrium

First look at an exchange economy without markets.

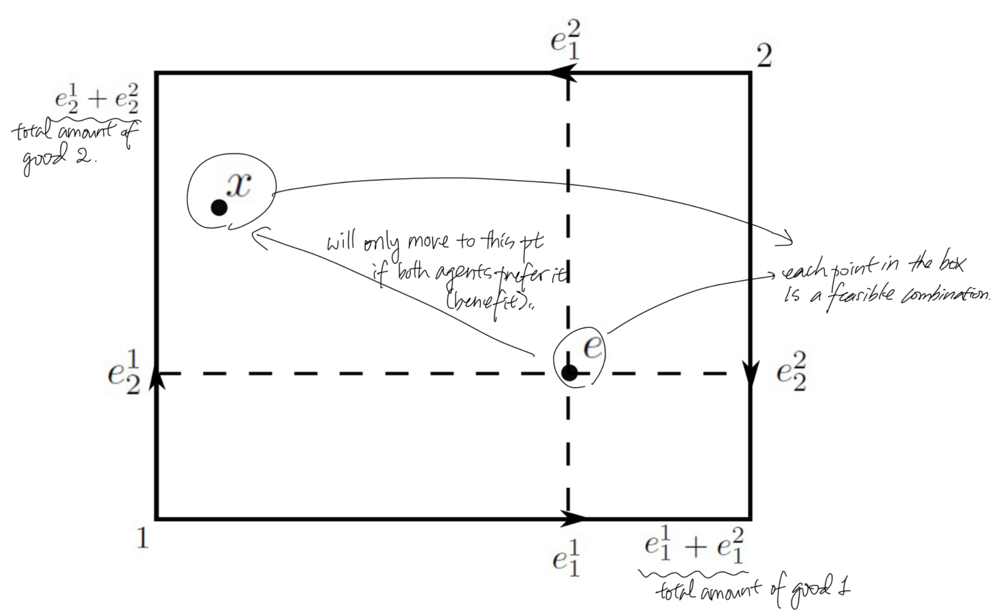
- no production
- agents have endowments
- private ownership is instituted (required for exchanges to happen, if not people can just steal)
- principle of voluntary, non-coercive trade is respected

The simplest case with two consumers and two goods:

$$e^1 = (e^1_1, e^1_2), \quad e^2 = (e^2_1, e^2_2)$$

Edgeworth box:

Figure 2: Edgeworth Box



Now extend to $I$ agents and $n$ goods. Suppose agents have complete, continuous, transitive, strictly convex preferences over bundles in $\mathbb{R}^n_+$.

## 3.1 Exchange Economy

(Barter) exchange economy: $\mathcal{E} = (\succsim_i, e^i)_{i \in I}$

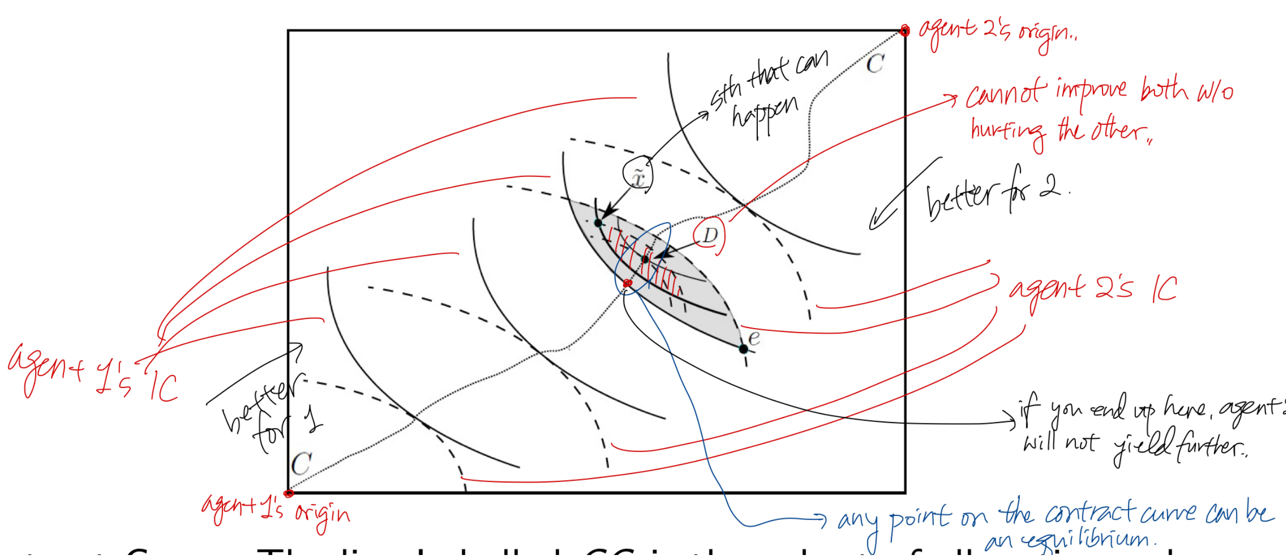
- $\{1, \ldots, I\}$: the set of agents (or indices for agents)
- endowment: $e \equiv (e^1, \ldots, e^I)$ with $e^i = (e^i_1, \ldots, e^i_n)$
- allocation: $x \equiv (x^1, \ldots, x^I)$
- $F(e) = \{x : \sum_i x^i = \sum_i e^i\}$: the set of feasible allocations given endowments.

**Definition 24** (Individual Rationality)**.** A feasible allocation is individually rational if $x^i \succsim_i e^i, \forall i$.

The allocation must be at least as good as my original endowment for an exchange to happen. If not satisfied, not rational (or not voluntary). Implication: trade must be mutually beneficial.

Contract Curve: subset of allocations where the consumers' indifference curves through the point are tangent to each other.

Figure 3: Contract Curve




The point $D$ on the contract curve is an equilibrium in a Barter exchange.

**Definition 25** (Pareto Optimal (PO))**.** A feasible allocation $x \in F(e)$ is Pareto efficient (or Pareto optimal) if there is no other feasible allocation such that $y^i \succsim x^i$ for all $i$ with at least one preference strict, i.e., $y^i \succ x^i$ for at least one $i$.

**Definition 26** (Blocking coalition)**.** Let $S \subset I$ denote a *coalition* of consumers. $S$ blocks $x \in F(e)$ if there is an allocation $y$ such that

1. $\sum_{i \in S} y^i = \sum_{i \in S} e^i$ (feasible within the coalition)
2. $y^i \succsim_i x^i$ for all $i \in S$ and $y^i \succ_i x^i$ for at least one $i$.

If $x$ is unblocked, it is PO. (The converse is not true.) A Pareto optimal allocation is an allocation that is unblocked with respect to the grand coalition (i.e., all agents).

**Definition 27** (Core of an exchange economy)**.** The core of an exchange economy with endowment $e$, denoted by $\mathcal{C}(e)$, is the set of all unblocked feasible allocations.
*Remark.* Core is a set: $\mathcal{C}(e)$.

- This notion of a core is a cooperative game-theoretic solution concept because it considers coalitions of players.
- Core is nonempty
- Core shrinks the larger the economy (more agents, more possibility of coalitions).

An allocation $x \in F(e)$ is an equilibrium in the exchange economy with endowment $e$ if $x$ is not blocked by any coalition of consumers, i.e., in the core.

Assumption 1: Each utility $u^i$ is continuous, strictly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

At prices $p$, the value of $i$th agent's endowment is: $p \cdot e^i$. This is the agent's income/wealth (now based on the price). Then, consumer $i$ solves the following maximization problem:

$$\max_{x \in \mathbb{R}^n_+} u^i(x)$$

subject to

$$p \cdot x = p \cdot e^i.$$

Under Assumption 1, the problem has a unique solution (the consumer's demand function, $x^i(p, p \cdot e^i)$). If $x^i_1 \geq e^i_1$, consumer $i$ is a buyer of good 1. If $x^i_1 \leq e^i_1$, consumer $i$ is a seller of good 1.

**Definition 28** (Excess demand)**.** Excess demand for good $k$ given prices $p \gg 0$ is defined as

$$z_k(p) = \sum_{i \in I} x^i_k(p, p \cdot e^i) - \sum_{i \in I} e^i_k,$$

where $\sum_{i \in I} x^i_k(p, p \cdot e^i)$ is the total demand of good $k$ and $\sum_{i \in I} e^i_k$ is the total supply.

Denote aggregate excess demand as $z(p) = (z_1(p), \ldots, z_n(p))$.

**Theorem 27** (Properties of Excess Demand). *If $u^i$ is continuous, strictly increasing and strictly quasiconcave for all $i$, then*

1. *$z(p)$ is continuous on $\mathbb{R}^n_{++}$*
2. *$z(p)$ is homogeneous of degree 0 in $p$*
3. *$z(p)$ satisfies Walras' law: $p \cdot z(p) = 0, \forall p \gg 0$.*

*Proof.*

1. Continuity follows from the continuity of individual demands.
2. Homogeneity of degree zero follows immediately from the budget constraint. $kp \cdot x^i = kp \cdot e^i \iff p \cdot x^i = p \cdot e^i$.
3. Budget balance, induced by monotonicity, implies $p \cdot x^i = p \cdot e^i$. Then, for agent $i$, we have

$$\sum_{k=1}^{n} p_k(x_k^i(p, p \cdot e^i) - e_k^i) = 0$$

Summing over $i$, we get

$$\sum_{i=1}^{I}\sum_{k=1}^{n} p_k(x_k^i(p, p \cdot e^i) - e_k^i) = 0 \Rightarrow \sum_{k=1}^{n} p_k \sum_{i=1}^{I}(x_k^i(p, p \cdot e^i) - e_k^i) = 0 \Rightarrow \sum_{k=1}^{n} p_k z_k(p) = 0$$

as desired.

□

With two goods, $p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) = 0, p_1, p_2 > 0$. $z_1(p) = 0 \Rightarrow z_2(p) = 0$. If we have $n-1$ markets clearing, the $n$th market also clears. All markets are interdependent through the price (i.e., price of one good affects the income of all agents).

**Definition 29** (Walrasian Equilibrium). A price vector $p^* \in \mathbb{R}^n_{++}$ is called a Walrasian equilibrium if $z(p^*) = 0$.

**Theorem 28** (Aggregate excess demand and existence of Walrasian equilibrium). *Suppose $z : \mathbb{R}^n_{++} \to \mathbb{R}^n$ has the following properties:*

1. *$z(\cdot)$ continuous on $\mathbb{R}^n_{++}$*
2. *$p \cdot z(p) = 0, \forall p \gg 0$ (Walras' law)*
3. *If $\{p^n\}$ is a sequence of price vectors in $\mathbb{R}^n_{++}$ converging to $\bar{p} \neq 0$ and $\bar{p}_k = 0$ for some $k$, then for some good $k'$ with $\bar{p}_{k'} = 0$, the associated sequence of excess demand for good $k'$,*

*$z_{k'}(p^n)$ is unbounded above.*

*Then, there is a price vector $p^* \gg 0$ such that $z(p^*) = 0$.*

Recall that $u^i$ continuous, strictly increasing, strictly quasiconcave on $\mathbb{R}^n_+$ implies $z(p)$: continuous and satisfies the Walras' law.

It would be nice if condition 3 in the above theorem is also satisfied.

**Theorem 29** (Utility and aggregate demand)**.** *If each consumer's utility is continuous, strictly increasing, strictly quasiconcave on $\mathbb{R}^n_+$ and if the aggregate endowment of each good is positive, then the aggregate excess demand satisfies conditions 1-3 for the existence theorem.*

*Proof.* The aggregate excess demand of good $k$ is

$$\sum_{i=1}^{n} x_i^k(p, p \cdot e^i) - \sum_{i=1}^{n} e_i^k$$

1. Continuity follows from the continuity of individual demands.
2. Walras' law follows from budget balance since each $u^i$ is strictly increasing.
3. Take a consumer that has positive income at $\bar{p} \neq 0$, so $\bar{p} \cdot e^i > 0$ ($\bar{p}$ is the limit vector in condition 3 of existence theorem). Take a sequence $p^m \to \bar{p}$. For a contradiction, suppose $x^m \equiv x_i^m(p^m, p^m \cdot e^i)$ is bounded. Then, there exists a convergent subsequence $x^m \to x^*$. Take $\hat{x}_i = x_i^* + (0, \ldots, 0, 1, 0, \ldots, 0)$ (i.e., 1 at good $k$ with the zero price).

   By our assumption, $\bar{p} \cdot e^i = \bar{p} \cdot x_i^* > 0$ because $\bar{p}_k = 0, \bar{p} \cdot e^i = \bar{p} \cdot x_i^* = \bar{p} \cdot \hat{x}_i$.

   Take $t\hat{x}, t \in (0, 1)$ close to 1. Then, because $u(\cdot)$ is continuous and strictly increasing, $u(t\hat{x}) > u(x^*)$, which is a contradiction as $x^*$ was utility maximizing.

□

Corollary: If each $u^i$ is continuous, strictly increasing, strictly quasiconcave on $\mathbb{R}^n_+$ and $\sum e^i \gg 0$, then there exists at least one price vector $p^*$ such that $z(p^*) = 0$.

## 3.2 Exchange Economy with Private Ownership

Now we add production. New issue: profits must be distributed. Firms: $\{1, \ldots, J\}$. Each firm has a production set $Y^j$, $y^j \in \mathbb{R}^n$ denotes firm $j$'s production plan. If $y_k^j > 0$, good $k$ is an output for firm $j$. If $y_k^j < 0$, good $k$ is an input for firm $j$.

Assumption 2:

1. $0 \in Y^j \subseteq \mathbb{R}^n$ (inaction, guarantees nonnegative profits)

2. $Y^j$ is closed and bounded (imposes continuity in production plans)
3. $Y^j$ is strongly convex, i.e., if $y_1, y_2 \in Y^j$, for all $t \in (0,1)$, there exists $\bar{y} \in Y^j$ such that

$$\bar{y} \geq ty_1 + (1-t)y_2$$

(rules out constant/increasing returns to scale and guarantees existence of profit-maximizing production plans.)

Firm $j$ solves

$$\max_{y^j \in Y^j} p \cdot y^j, \forall p \gg 0.$$

$\pi^j(p) = \max_{y^j \in Y^j} p \cdot y^j$ (profit = value function of profit maximization)

**Theorem 30** (Properties of the supply and profit functions)**.** *If $Y^j$ is closed and bounded, strongly convex and $0 \in Y^j$, then $\forall p \gg 0$, the solution of the firm's problem is unique and denoted by $y^j(p)$, which is continuous on $\mathbb{R}^n_{++}$. In addition, $\pi^j(p)$ is continuous on $\mathbb{R}^n_{++}$ and well defined.*

(Essentially follows from the theorem of the maximum.)

Aggregate production possibility set:

$$Y = \left\{ \sum_{j \in J} y^j : y^j \in Y^j \right\}.$$

**Theorem 31.** *If each $Y^j$ is closed and bounded, strongly convex and $0 \in Y^j$, $Y$ also satisfies the same properties.*

**Theorem 32** (Aggregate Profit Maximization)**.** *For any prices $p \geq 0$, we have $p \cdot \bar{y} \geq p \cdot y, \forall y \in Y$ if and only if for some $\bar{y}^j \in Y^j, j \in J$, we may write $\bar{y} = \sum_j \bar{y}^j$ and $p \cdot \bar{y}^j \geq p \cdot y^j, \forall y^j \in Y^j, j \in J$.*

*Proof.*

($\Rightarrow$) Take $\bar{y} \in Y$ that maximizes aggregate profit. Suppose $\bar{y} = \sum_{j \in J} \bar{y}^j$ for $\bar{y}^j \in Y^j$. If $\bar{y}^k$ does not maximize profits for firm $k$, this means there exists $\tilde{y}^k \in Y^k$ such that $p \cdot \tilde{y}^k > p \cdot \bar{y}^k$. Then, take $\tilde{y} \in Y$ such that $\tilde{y} = \tilde{y}^k + \sum_{j \neq k} \bar{y}^k$. Then,

$$p \cdot \left( \tilde{y}^k + \sum_{j \neq k} \bar{y}^j \right) > p \cdot \bar{y},$$

which is a contradiction.

($\Leftarrow$) Suppose $\bar{y}^1, \ldots, \bar{y}^J$ maximize individual profits. Then, $\sum_{j \in J} p \cdot \bar{y}^j \geq \sum_{j \in J} p \cdot y^j, \forall y^j \in$

$Y^j, \forall j \in J$.

$$\Rightarrow p \cdot \left(\sum_{j \in J} \bar{y}^j\right) \geq p \cdot \left(\sum_{j \in J} y^j\right) \iff p \cdot \bar{y} \geq p \cdot y, \forall y \in Y.$$

□

- Consumers: $\{1, \ldots, I\}$ (consume nonnegative amount of goods)
- Private ownership: $0 \leq \theta_{ij} \leq 1$ denotes person $i$'s share in firm $j$

Budget constraint of person $i$ becomes

$$p \cdot x^i \leq \underbrace{p \cdot e^i + \sum_{j \in J} \theta_{ij} \pi_j(p)}_{m^i(p)}$$

Utility maximization:

$$\max_{x^i \in \mathbb{R}^n_+} u^i(x^i)$$

subject to

$$p \cdot x^i \leq m^i(p)$$

$m^i(p) \geq 0$ whenever $p \geq 0, e^i \geq 0, \pi_j(p) \geq 0, \forall j$.

**Theorem 33** (Basic properties of demand with profit shares)**.** *If $Y^j$ is closed and bounded, strongly convex and $0 \in Y^j$ and $u^i$ is continuous, strictly increasing and strictly quasiconcave, then the solution to the utility maximization problem exists and is unique, $\forall p \gg 0$. Also, $x^i(p, m^i(p))$ is continuous in $p \in \mathbb{R}^n_{++}$ and $m^i(p)$ is continuous.*

**Theorem 34** (Existence with Production)**.** *Consider the economy $(u^i, e^i, \theta_{ij}, Y^j)$ with $i \in I, j \in J$. If $u^i$ is continuous, strictly increasing and strictly quasiconcave and $Y^j$ is closed and bounded, strongly convex and $0 \in Y^j$ and $y + \sum_i e^i \gg 0$ for some production vector $y = \sum_j y^j$, then there exists at least one $p^* \gg 0$ such that $z(p^*) = 0$.*

*Proof.* Verify that $z(\cdot)$ satisfies the three properties of our existence theorem.

$$z_k(p) = \sum_{i \in I} x^i_k(p, m^i(p)) - \sum_{j \in J} y^j_k(p) - \sum_{i \in I} e^i_k$$

1. $z(p)$ is continuous by the continuity of $x^i_k, y^j_k, \forall k, i$, (theorem of the maximum).
2. Walras law: same as before.
3. It suffices to show that there exists a consumer with strictly positive income at the limit

vector $p$.

Because $y + \sum_{i \in I} e^i \gg 0$ for some $y$ and $\bar{p} \neq 0$, then $\bar{p} \cdot \left(y + \sum_{i \in I} e^i\right) > 0$.

$$\begin{aligned}
\sum_{i \in I} m^i(p) &= \sum_{i \in I} \left( \bar{p} \cdot e^i + \sum_{j \in J} \theta_{ij} \pi_j(p) \right) \\
&= \sum_{i \in I} \bar{p} \cdot e^i + \sum_i \sum_j \theta_{ij} \pi_j(p) \\
&= \bar{p} \sum_{i \in I} e^i + \sum_j \sum_i \theta_{ij} \pi_j(p) \\
&= \bar{p} \sum_{i \in I} e^i + \sum_j \pi_j(p) \\
&\geq \sum_i (\bar{p} \cdot e^i + \bar{p} \cdot y) \\
&= \bar{p} \sum_i (e^i + y) > 0
\end{aligned}$$

We are done, as $\sum_i m^i(p) > 0$, each $m^i(p) \geq 0$, there exists $i \in I$ such that $m^i(p) > 0$.

□

**Definition 30** (Walrasian Equilibrium Allocation (WEA))**.** An allocation $(x^*, y^*)$ and a price vector $p^*$ form a Walrasian equilibrium if

1. profits are maximized: For each $j \in J, p^* \cdot y_j^* \geq p^* \cdot y_j, \forall y_j \in Y^j$
2. consumer choices are optimal: For each $i \in I, x_i^* \succsim x_i, \forall x_i \in B_i(p^*)$.
3. markets clear: $\sum_i x_i^*(p) = \sum_i e^i + \sum_j y_j^*(p) (\Rightarrow z(p^*) = 0)$

## 3.3 Welfare Properties of Walrasian Equilibrium

Recall that Pareto optimal allocation: A feasible allocation $(x, y)$ is Pareto optimal if there is no feasible allocation $(x', y')$ such that $x_i' \succsim_i x_i, \forall i \in I$ and $x_j' \succ_j x_j$ for some $j$.

From here on, we take $\succsim_i, i \in I$ to be a rational preference relation.

Local nonsatiation (revisted):
A preference relation $\succsim_i$ on $X_i$ is satiated at $y$ if there is no $x \in X_i$ such that $x \succ_i y$.

In our setting: $X_i \subseteq \mathbb{R}_+^n$. A preference relation $\succsim_i$ on $X_i \subseteq \mathbb{R}_+^n$ is locally nonsatiated if for every $x \in X_i$ and $\varepsilon > 0$, there exists $x' \in X_i$ such that $\|x' - x\| \leq \varepsilon$ and $x' \succ_i x$.

Local nonsatiation gives the following:

**Lemma.** Suppose $\succsim_i$ is locally nonsatiated and $x_i^*$ is defined as follows:

$$x_i^* \succsim_i x_i, \forall x_i \in B_i(p) := \{x_i : p \cdot x_i \leq m_i(p)\}.$$

Then, $x_i \succsim_i x_i^*$ implies $p \cdot x_i \geq m_i(p)$ and $x_i \succ_i x_i^*$ implies $p \cdot x_i > m_i(p)$.

*Proof.* (This was a part of Problem Set 10).

For a contradiction, suppose the second part of the lemma does not hold. Then, there exists $x^i$ such that $x^i \succ_i x^{i*}$ and $p \cdot x^i \leq m_i(p)$, which is a blatant contradiction to the fact that $x^{i*} \succsim_i x^i, \forall x^i \in \{x^i : p \cdot x^i \leq m_i(p)\}$.

Now, suppose the first part of the lemma does not hold. Then, therest exists $x^i$ such that $x^i \succsim_i x^{i*}$ and $p \cdot x^i < m_i(p)$. Consider an $\varepsilon > 0$ small enough such that $p \cdot (x^i + \varepsilon \mathbb{1}_n) \leq m_i(p)$, where $\mathbb{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^n$. By local nonsatiation, there exists $\tilde{x}^i$ such that $\|x^i - \tilde{x}^i\| \leq \varepsilon$ and $\tilde{x}^i \succ_i x^i$. By transitivity, $\tilde{x}^i \succ_i x^{i*}$ and $p \cdot \tilde{x}^i \leq m_i(p)$, which is a contradiction.

Hence, the lemma must be true. □

Recall that the total wealth in the economy is

$$\sum_i m_i(p) = \sum_i \left( p \cdot e^i + \sum_j \theta_{ij} p \cdot y_j^*(p) \right)$$

**Theorem 35** (First Welfare Theorem)**.** *Suppose that each consumer's preference is locally nonsatiated. Then, any allocation $(x^*, y^*)$ that forms a WEA with price vector $p^*$ is Pareto optimal*

*Proof.* For a contradiction, suppose $(x^*, y^*)$ is WEA but not PO. Then, there exists a feasible allocation $(x, y)$ such that $x_i \succsim_i x_i^*, \forall i$ and $x_i \succ x_i^*$ for some $i$.

Local nonsatiation implies

$$x_i \succsim_i x_i^* \Rightarrow p^* \cdot x_i \geq p^* \cdot e^i + \sum_j \theta_{ij}(p^* \cdot y_j^*), \forall i \in I$$

and

$$x_i \succ_i x_i^* \Rightarrow p^* \cdot x_i > p^* \cdot e^i + \sum_j \theta_{ij}(p^* \cdot y_j^*)$$

for some $i \in I$. Summing over $i$ yields:

$$\sum_i p^* \cdot x_i > \sum_i p^* \cdot e^i + \sum_i \sum_j \theta_{ij} p^* \cdot y_j^* = \sum_i p \cdot e^i + \sum_j p^* \cdot y_j^*$$

Since $y_j^*$ maximizes profits,

$$\sum_j p^* \cdot y_j^* \geq \sum_j p^* \cdot y_j, \forall y_j \in Y^j.$$

Combining the above two equations gives

$$\sum_i p^* \cdot x_i > \sum_i p^* \cdot e^i + \sum_j p^* \cdot y_j^* \geq \sum_j p^* \cdot e^i + \sum_j p^* \cdot y_j,$$

which is contradictory to the fact that $(x, y)$ is feasible, i.e., $\sum_i x_i = \sum_i e^i + \sum_j y_j$. □

**Theorem 36.** *Any competitive equilibrium allocation is in the core.*

(This was a part of Problem Set 10.)

*Proof.* For a contradiction, suppose not. Then, there exists a WEA $(x^*, y^*)$ and a price vector $p^*$ that is not in the core. This implies that there exists a blocking coalition $S$ for this allocation. That is, there exists $x = (x^1, x^2, \ldots, x^I)$ such that

1. is feasible, i.e.,

$$\sum_{i \in S} p^* \cdot x^i = \sum_{i \in S} m_i(p^*)$$

2. $x^i \succsim_i x^{i*}, \forall i \in S$ and $x^i \succ_i x^{i*}$ for at least one $i$.

For this particular $i$ with strict preference, we have

$$p^* \cdot x^i > m_i(p^*)$$

implied by local nonsatiation (as shown in the previous problem). For other $i \in S$,

$$p^* \cdot x^i \geq m_i(p^*),$$

which is, again, implied by local nonsatiation. Then,

$$\sum_{i \in S} p^* \cdot x^i \geq p^* \cdot x^i + \sum_{k \in S} p^* \cdot x^{k*}(p^*) > \sum_{i \in S} x^{i*}(p^*) = \sum_{i \in S} m_i(p^*),$$

i.e., the allocation is not feasible within the coaltion, which is a contradiction. Hence, there can be no blocking coalition and, in turn, WEA must be in the core. □

*Remark.*

- The set of core allocations is typically much larger than the set of WEA.

- K-replica economies (many agents of the same type, a market microstructure concept)
- The core "shrinks" in large replica economies ($k \to \infty$, core allocations converge to WEA).

Now we would like to go the other way: start with a PO allocation and find a price vector that supports it as a WEA (= the Second Welfare Theorem). This direction is more restrictive and thus we need more concepts.

**Definition 31** (Walrasian equilibrium with transfers)**.** Given an economy $(\{X_i, \succsim_i\}_{i\in I}, (\theta_{ij}), \{Y^j\}_{j\in J}, e)$, an allocation $x^*, y^*$ and a price vector $p^*$ constitute an equilibrium with transfers if there exists a price vector $p^*$ and wealth levels $w = (w_1, \dots, w_I)$ with

$$\sum_i w_i = p^* \cdot e + \sum_j p^* \cdot y_j$$

such that

1. firms maximize profits: $p^* \cdot y_j^* \geq p^* \cdot y_j, \forall y_j \in Y^j, j \in J$
2. consumer choices are optimal: $x_i^* \succsim_i x_i, \forall x_i : p^* \cdot x_i \leq w_i$
3. markets clear (or is feasible): $\sum_i x_i^*(p^*) = \sum_i e^i + \sum_j y_j^*(p^*)$
4. transfers are balanced: $T_i = w_i - (p^* \cdot e + \sum_j \theta_{ij} y_j^*)$

Convexity implies the existence of a hyperplane that supports consumer's "better-than-set".

**Definition 32** (Quasi-Equilibrium)**.** Given an economy $(\{X_i, \succsim_i\}_{i\in I}, \{Y^j\}_{j\in J}, \theta_{ij}, e)$, an allocation $x^*, y^*$ and a price vector $p^*$ constitute a quasi-equilibrium with transfers $w \in (w_1, \dots, w_I)$ with $\sum_i w_i = p^* \cdot e + \sum_j p^* \cdot y_j^*$ if it satisfies the following:

1. profit maximization: $\forall j \in J, p^* \cdot y_j^* \geq p^* \cdot y_j, \forall y_j \in Y^j$.
2. consumer "optimality": $\forall i \in I, x \succ_i x_i^*$ implies $p^* \cdot x \geq w_i$ (instead of $>$ in an usual equilibrium)
3. feasibility (market clearing): $\sum_i x_i^* = \sum_i e^i + \sum_j y_j^*$
4. transfers $T_i$ balance: $\sum_i T_i = \sum_i [w_i - (p^* \cdot e + \sum_j \theta_{ij} p^* \cdot y_j^*)] = 0$

**Theorem 37** (Second Welfare Theorem)**.** *Consider an economy $(\{X_i, \succsim_i\}_{i\in I}, \{Y^j\}_{j\in J}, \theta_{ij}, e)$ and assume that $Y^j$ is convex for all $j \in J$ and $\succsim_i$ is rational, convex and locally nonsatiated, $\forall i \in I$. Then, for each Pareto optimal allocation $(x^*, y^*)$, there is a price vector $p^* \neq 0$ such that $(x^*, y^*)$ together with $p^*$ form a quasi-equilibrium with transfers.*

*Proof.* The proof uses the separating hyperplane theorem. If an allocation is PO, there is a hyperplane that simultaneously supports the better-than-set of al consumers and all producers. This hyperplane yields a candidate equilibrium vector. The proof consists of three parts:

1. aggregation (sum up)
2. separation (find separating hyperplane)
3. decentralization (check the price maximizes, very important concept!)

1. **Aggregation**

   First, we aggregate all consumer's preference when evaluating the Pareto optimal consumption bundle $x^*$. Define the following set:

$$V_i := \{x_i \in X_i : x_i \succ_i x_i^*\} \subset \mathbb{R}^n.$$

   Then, let $V = \sum_{i \in I} V_i$. Claim: $V$ is convex.

   We show that each $V_i$ is convex. Take $x', x'' \in V_i$. Without loss of generality, assume $x' \succsim_i x''$. Because preferences are convex, for any $\lambda \in [0,1]$, $\lambda x' + (1-\lambda)x'' \succsim_i x'' \succ_i x_i^*$. Therefore, $\lambda x' + (1-\lambda)x'' \in V_i$ and thus $V_i$ is convex. Then, $V$ is convex, as it is a finite sum of convex sets.

   Now we aggregate producers. Define

$$Y = \sum_j Y^j = \{y \in \mathbb{R}^n : \sum_j y_j \in \mathbb{R}^n, y_j \in Y^j, \forall j \in J\}.$$

   The set of consumption bundles that can be allocated to consumers is $Y + e$, which is convex because it is the sum of $J+1$ convex sets.

2. **Separation**

   Next, we separate the sets $V$ and $Y + e$. Since $(x^*, y^*)$ is PO, $V \cap (Y + e) = \varnothing$. Otherwise, there exists a feasible consumption bundle that would make at least one consumer strictly better and everyone else weakly better off. This would be a contradiction to the fact that $(x^*, y^*)$ is PO.

   Because $V, Y + e$ are two disjoint, convex sets, we can apply the separating hyperplane theorem. By this theorem, there exists $p \in \mathbb{R}^n$ with $p \neq 0$ and $r \in \mathbb{R}$ such that

$$p \cdot z \geq r, \forall z \in V, \quad p \cdot z \leq r, \forall z \in Y + e.$$

   For consumers, we claim: If $x_i \succsim_i x_i^*$, then $p \cdot \sum_i x_i \geq r$.

   Take any $x_i \succsim_i x_i^*, \forall i$. By local nonsatiation, for each $i$, there exists $\hat{x}_i$ near $x_i$ such that $\hat{x}_i \succ_i x_i$. Hence, $\hat{x}_i \in V_i, \forall i, \sum_i \hat{x}_i \in V$. So $p \cdot \sum_i \hat{x}_i \geq r$. Take a sequence of $\hat{x}_i$ such that

$\hat{x}_i \to x_i, (\hat{x}_i^n = x_i + \frac{1}{n})$.

For every $k$, $\sum_i \hat{x}_i^k \in V, p \cdot \sum_i \hat{x}_i^k \geq r$. $\Rightarrow p \cdot \sum_i \hat{x}_i^k \to p \cdot \sum_i x_i^* \geq r$ as $k \to \infty$.

We have shown that $\sum_i x_i^*$ belongs to the closure of $V$, which in turn is contained in the half space $\{z \in \mathbb{R}^n : p \cdot z \geq r\}$.

For producers, we have

$$p \cdot z \leq r, \forall z \in Y + e$$

Set $z = \sum_j y_j^* + e$, then we have that

$$p \cdot \left( \sum_j y_j^* + e \right) \leq r$$

Putting the two implications together and using the fact that $(x^*, y^*)$ is PO (PO allocation is feasible), thus

$$\sum_i x_i^* = \sum_j y_j^* + e \in Y + e.$$

Therefore, we must have $p \cdot (\sum_i x_i^*) = r$.

3. **Decentralization**

Claim: $x^*$ satisfies the consumer's condition in quasi-equilibrium with $p^* = p$.

For some consumer $i$, take $x$ such that $x \succ_i x_i^*$. We need to show $p^* \cdot x \geq w_i$.

$$p^* \cdot \left( x + \sum_{j \neq i} x_j^* \right) \geq r \geq p^* \cdot \left( x_i^* + \sum_{j \neq i} x_j^* \right) \Rightarrow p^* \cdot x \geq p^* \cdot x_i^*.$$

We also know that, under local nonsatiation, budget constraint binds and therefore $p^* \cdot x_i^* = w_i$. Hence, $p^* \cdot x \geq w_i$, and hence we have the claim.

Claim: $y^*$ maximizes profits at prices $p^*$.

For any firm $j$ and $y_j \in Y^j$, we have $y_j + \sum_{k \neq j} y_k^* \in Y$. By separation,

$$p^* \cdot \left( e + y_j + \sum_{k \neq j} y_k^* \right) \leq r = p^* \cdot \left( e + y_j^* + \sum_{k \neq j} y_k^* \right) \Rightarrow p^* \cdot y_j \leq p^* \cdot y_j^*, \forall j \in J.$$

Therefore, $y_j^*$ maximizes profits at prices $p^*$. We have shown that PO allocation $(x^*, y^*)$ and price vector $p^*$ form a quasi-equilibrium with transfers.

□

When is a quasi-equilibrium an actual equilibrium? One can show that, under local nonsatiation, if there is a consumption bundle cheaper than the consumer's wealth, then a quasi-equilibrium is a WEA

$$x_i \succ_i x_i^* \Rightarrow p \cdot x_i \succ_i w_i.$$

Recall: Pareto optimal allocations.

An allocation is feasible if and only if

$$\sum_{i \in I} x^i \leq \sum_{i \in I} e^i + \sum_{j \in J} y^j.$$

Consider an economy with $I$ consumers. A feasible allocation is Pareto optimal if there is no other feasible allocation $\tilde{x}$ such that $u^i(\tilde{x}^i) \geq u^i(x^i)$ for all $i \in I$, $u^i(\tilde{x}^i) > u^i(x^i)$ for some $i$.

**Definition 33** (Utility possibility set)**.** Define

$$U(x) := \{u^i\}_{i \in I} \text{ s.t. } x\text{: feasible}$$

The utility possibility set is defined as

$$\mathcal{U} = \{(u^1, \ldots, u^I) \in \mathbb{R}^I : u^i(x^i) \geq u^i, \forall i \in I, x \text{ feasible}\}.$$

$$\mathcal{U}_{-1} = \{(u^2, \ldots, u^I) \in \mathbb{R}^{I-1} : (u^1, \ldots, u^I) \in \mathcal{U} \text{ for some } u^1 \in \mathbb{R}\}.$$

An alternative definition of Pareto optimality.
A feasible allocation $x^*$ is Pareto optimal if the set of $\{u \in \mathcal{U} : u \geq u^*\} = \{u^*\}$ where $u^* = (u^1(x^{1*}), \ldots, u^I(x^{I*}))$.

Let $\partial\mathcal{U}$ denote the boundary of the utility possibility set $\mathcal{U}$. If $u \in \partial\mathcal{U}$, $\nexists u' \in \mathcal{U}$ such that $u'_i \geq u_i, \forall i$ and $u'_i > u_i$ for some $i$.

Claim: A feasible allocation $x^*$ is PO if and only if $u^* = (u^1(x^{1*}), \ldots, u^I(x^{I*})) \in \partial\mathcal{U}$.

*Proof.*

($\Rightarrow$) Let $x^*$ be PO and suppose $u^* \notin \partial\mathcal{U}$. Then, $\exists \bar{u} > u^*$ which implies that $u^*, \bar{u} \in \{u \in \mathcal{U} : u \geq u^*\}$. This contradicts the PO of $x^*$.

($\Leftarrow$) Let $u^* \in \partial\mathcal{U}$ and suppose it is not PO. Then, there exists a feasible allocation $\tilde{x}$ such that $u^i(\tilde{x}^i) \geq u^{i*}, \forall i$ and $u^i(\tilde{x}^i) > u^{i*}$ for some $i$. Then, $u^* \notin \partial\mathcal{U}$, which is a contradiction.

□

Note that we are using $\partial\mathcal{U}$ as a synonym of the utility possibility frontier.

**Proposition 6.** Let the utility function $u^i, i \in I$ be continuous and $u^1$ be strictly increasing. Then, $x^*$ is PO if and only if it is a solution to the following program:

$$\max_{x^1} u^1(x^1)$$

subject to

$$u^i(x^i) \geq \bar{u}^i, \forall i = 2, \ldots, I, \sum_i x^i \leq \sum_i e^i + \sum_j y^j, x^i \geq 0, \forall i \in I$$

for some $(\bar{u}^2, \ldots, \bar{u}^I) \in \mathcal{U}_{-1}$.

*Proof.*

($\Rightarrow$) A PO allocation $x^*$ solves the program. By definition, if we set $\bar{u}^i = u^{i*}, \forall i = 2, \ldots, I$, then $u^{1*}$ is the maximum utility that we can give to person 1, so $x^*$ is a solution.

($\Leftarrow$) A solution to the above maximization is PO. Let $x^*$ solve the above program and suppose for a contradiction that it is not PO. Then, there exists $\tilde{x} \neq x^*$ such that $\tilde{x}$ is feasible and $u^i(\tilde{x}^i) \geq u^i(x^{i*})$ for some $i$. Now suppose $u^k$ with $k \neq 1$ is such that

$$u^k(\tilde{x}^k) > u^k(x^{*k})$$

Case 1. Suppose $\tilde{x}^k > 0$. Without loss of generality, $\tilde{x}^k_1 \neq 0$. $\tilde{x}^k = (\tilde{x}^k_1, \ldots \tilde{x}^k_n)$. Let $w \in \mathbb{R}^n$ such that $w^1 = 1, w^\ell = 0, \forall \ell = 2, \ldots, n$. $w = (1, 0, \ldots, 0)$.

By continuity of $u^k$, there exists $\varepsilon > 0$ such that $u^k(\tilde{x}^k - \varepsilon w) > u^k(x^{k*})$. Consider $\hat{x}$ such that $\hat{x}^1 = \tilde{x}^1 + \varepsilon w, \hat{x}^i = \tilde{x}^i, \forall i \neq 1, k, \hat{x}^k = \tilde{x}^k - \varepsilon w$.

Note that $\hat{x}$ is feasible, since $\tilde{x}$ is feasible (just reallocated a little from $k$ to 1). By strict monotonicity of $u^1$, we have

$$u^1(\hat{x}^1) > u^1(\tilde{x}^1) \geq u^1(x^{*1}).$$

Note also that everyone else is getting at least as much utility as at $x^{i*}$, i.e.,

$$u^i(\hat{x}^i) = u^i(\tilde{x}^i) \geq u^i(\tilde{x}^i) \geq u^i(x^{*i}) \geq \bar{u}^i, \forall i = 2, \ldots, I.$$

This contradicts the maximality of $x^*$ in the above program, i.e., there exists a feasible candidate for the program that strictly increases the objective function.

Case 2. Suppose $\tilde{x}^k = 0$. We know $u^k(\tilde{x}^k) > u^k(x^{*k})$. We also know that $x^{*k} \geq 0$ as a constraint in the above program. Now consider

$$\hat{x}^1 = x^{*1} + x^{*k}, \quad \hat{x}^k = \tilde{x}^k = 0, \quad \hat{x}^i = x^{*i}, \forall i \neq 1, k.$$

Then, $u^i(\hat{x}^i) = u^i(\tilde{x}^i) \geq u^i(x^{*i}), \forall i = 2, \dots, I$ and $u^1(\hat{x}^1) > u^1(x^{*1})$ since person 1's utility is strictly increasing.

□

Let $\lambda \in \mathbb{R}^I_{++}$ and consider the following planner's problem:

$$\max_x \mathcal{U}_p(x) = \sum_{i \in I} \lambda^i u^i(x^i)$$

subject to

$$\sum_{i \in I} x^i \leq \sum_{i \in I} e^i + \sum_{j \in J} y^j, x^i \geq 0, \forall i$$

**Proposition 7** (Sufficient condition for Pareto optimality)**.** If $x^*$ is a solution to the planner's problem, then $x^*$ is Pareto optimal.

*Proof.* Let $x^*$ be a solution to the planner's problem and suppose it is not PO. Then, there exists $\tilde{x} \neq x^*$ such that $\tilde{x}$ is feasible and $u^i(\tilde{x}^i) \geq u^i(x^{*i}), \forall i \in I$ and $u^i(\tilde{x}^i) > u^i(x^{*i})$ for some $i$.

Multiply each inequality by the corresponding $\lambda^i$ and sum over $i$:

$$\sum_{i \in I} \lambda^i u^i(\tilde{x}^i) > \sum_{i \in I} \lambda^i u^i(x^{*i}).$$

Hence, $x^*$ is not a solution, which is a contradiction. □

The converse is not true in general. What is additionally needed?

**Proposition 8.** Let $u^i$ be concave for each $i$ and $x^*$ be Pareto optimal. Then, there exists $\lambda \in \mathbb{R}^I_+ \setminus \{0\}$ such that $x^*$ is a solution to the planner's problem.

**Lemma.** Let $u^i$ be concave. Then, the utility possibility set is convex

*Proof.* Let each $u^i$ be concave. Then, for feasible bundles $x^{1i}, x^{2i}$ and $\alpha \in [0, 1]$, we have

$$u^i(\alpha x^{1i} + (1-\alpha)x^{2i}) \geq \alpha u^i(x^{1i}) + (1-\alpha)u^i(x^{2i}) \geq \alpha u^i + (1-\alpha)u^i = u^i.$$

Since this holds for any $i$, $\alpha u + (1-\alpha)u' \in \mathcal{U}$ for any $u, u' \in \mathcal{U}$. Thus, $\mathcal{U}$ is convex. □

**Lemma** (Separating Hyperplane Theorem)**.** Let $Z \subset \mathbb{R}^n$ be convex and $a \in \mathbb{R}^n$ not in the interior of $Z$. Then, there exists $p \in \mathbb{R}^n \setminus \{0\}$ such that $p \cdot a \geq p \cdot z, \forall z \in Z$

*Proof of Proposition.* Let $x^*$ be PO. Then, $u^* \in \partial\mathcal{U}$, so it is not in the interior. $\mathcal{U}$ is convex by the first lemma because $u^i$ is concave. Therefore, by the separating hyperplane theorem, there exists $\lambda \in \mathbb{R}^I \setminus \{0\}$ such that

$$\lambda \cdot u^* \geq \lambda \cdot u, \forall u \in \mathcal{U}.$$

Hence, $\sum_i \lambda^i u^{*i} \geq \sum_i \lambda^i u^i \;\; (*)$.

It remains to show that $\lambda^i \geq 0, \forall i \in I$. Suppose not. Then, there exists $k$ such that $\lambda^k < 0$. Then,

$$\lambda \cdot u^* \geq \sum_{i \neq k} \lambda^i u^i + \lambda^k u^k, \forall u \in \mathcal{U}$$

However, then $(*)$ cannot hold, as we can infinitely decrease $u^k$ to increase the sum. □

# 4 Cooperative Game Theory

Game theory can be thought as interactive decision theory. A central feature of multiperson interactions is the potential of strategic interdependence. There are two branches of game theory.

1. Non-cooperative game theory: focuses on a single player and studies what they can do to win the game
   ex. patent races, oligopoly pricing, bargaining, political competition.
2. Cooperative game theory: focus is on coalitions of players, design matching algorithms for market without transfers (medical residents, human organs, public housing, school seats)
   Used to design "fair" allocation procedures. Less popular and understudied.

Usual description of a game:

- set of players $I$
- set of outcomes $A$
- strategy profiles $S = X^{S_i}, i \in I$
- payoffs

Players have preferences over outcomes in set $A$. We assume that the preferences satisfy all axioms so that there is an expected utility representation (on which most game theory relies).

In cooperative game theory, a game is described in characteristic form. This is a summary of the payoffs available to each group of players in a context where binding agreements among players of the group are possible.

We can derive a characteristic description from a normal form game but actually the "reduced form" parsimonious description of a cooperative game has proven to be analytically powerful (i.e., Nash bargaining).

Set of players $I = \{1, \ldots, I\}$. Nonempty subsets of $I$, $S, T \subset I$: coalitions. An outcome is a list of utilities: $u = (u_1, \ldots, u_I) \in \mathbb{R}^I$. Given $u$, $u^S = (u_i)_{i \in S} \in \mathbb{R}^S$ is coalition S's utilities. $u^S$ is essentially the projection of $u \in \mathbb{R}^I$ to coordinates corresponding to $S$.

**Definition 34.** A nonempty closed set $U^S \subset \mathbb{R}^S$ is a utility possibility set for a coalition $S \subset I$ if it is comprehensive, i.e.,

$$u^S \in U^S, u'^S \leq u^S \Rightarrow u'^S \in U^S$$

**Definition 35.** A game in characteristic form $(I, V)$ is a set of players $I$ and a rule $V(\cdot)$ that associates to everyone in coalition $S \subset I$ a utility possibility set $V(S) \subseteq \mathbb{R}^S$, i.e., $V(S)$ is the payoffs players in $S$ can achieve.

**Example 8.**

1. Economy

   Consider an economy with $I$ consumers having continuous, increasing, concave utility functions $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ and endowments $w_i \geq 0$.

   Suppose there is publicly available technology $Y \subseteq \mathbb{R}^L$ that is convex and has constant returns to scale.

   Then, we can define a game in characteristic form by the letting

   $$V(S) = \left\{ (u^i(x^i))_{i \in S} : \sum_{i \in S} x_i = \sum_{i \in S} w_i + y, y \in Y \right\} - \mathbb{R}^S_+$$

   $V(S)$ is a set of utility vectors consumers in coalition $S$ can achieve by trading among themselves and using technology $Y$. Every set $V(S)$ is convex.

2. Majority voting

   Consider $I = 3$. 2 out of 3 form a majority. $A$: alternatives, $u^i(a) \geq 0, i = 1, 2, 3$.

   $$V(I) = \{u^1(a), u^2(a), u^3(a) : a \in A\} - \mathbb{R}^3_+$$

   $$V(\{i, h\}) = \{(u^i(a), u^h(a)) : a \in A\} - \mathbb{R}^{\{i,h\}}_+, \forall \{i, h\} \subseteq \{1, 2, 3\}$$

   i.e., any pair of players form a majority and thus can choose any alternative they want.

   $$V(\{i\}) = -\mathbb{R}_+$$

i.e., a single player cannot choose anything.

**Definition 36.** A game in characteristic form $(I, V)$ is superadditive, if for any coalitions $S, T \subset I$ that are disjoint (i.e., $S \cap T = \varnothing$), we have if $u^S \in V(S)$ and $u^T \in V(T)$, $u^S, u^T) \in V(S \cup T)$. Coalitions $S, T$ are able to do at least as well acting together rather than separately.

Setting with transferable utility:
For a situation described by a game in characteristic form, quasilinearity or transferable utility hypothesis amounts to saying that $V(S)$ are half spaces (i.e., sets whose boundaries are hyperplanes in $\mathbb{R}^S$). Choose units of utility so that the hyperplane defining $V(S)$ have normal vectors $(1, \ldots, 1) \in \mathbb{R}^S$.

$$V(S) = \{u^S \in \mathbb{R}^S : \sum_{i \in S} u_i^S \leq v^*(S)\}$$

for some $v^*(S) \in \mathbb{R}$. We can view the coalition as choosing to maximize total utility denoted by $v^*(S)$, which can be allocated to members of $S$.

**Definition 37.** A transferable utility game in characteristic form, TU-game, is defined by $(I, V)$ where $I$ is a set of players and $V(\cdot)$ is a function called the characteristic function that assigns to every coalition $S \subset I$ a number $v^*(S)$ called the worth of $S$.

TU-voting majority

$$v^*(I) = 3, \quad v^*(\{i\}) = 0, i = 1, 2, 3 \quad v^*(\{i, j\}) = 3, \forall i, j \in \{1, 2, 3\}.$$

The core: This is the set of utility possible outcomes with the property no coalition can improve on its own. An empty core is indicative of competitive instability.

**Definition 38.** Given a game in characteristic form $(I, V)$, the utility outcome $u \in \mathbb{R}^I$ is blocked or improved upon by a coalition $S \subset I$ if there exists $u'^S \in V(S)$ such that $u_i^S < u_i'^S, \forall i \in S$. If there is a TU-game $(I, v^*)$, then the outcome $u = (u_1, \ldots, u_I)$ is blocked by $S$ if and only if $\sum_i u_i < v^*(S)$.

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